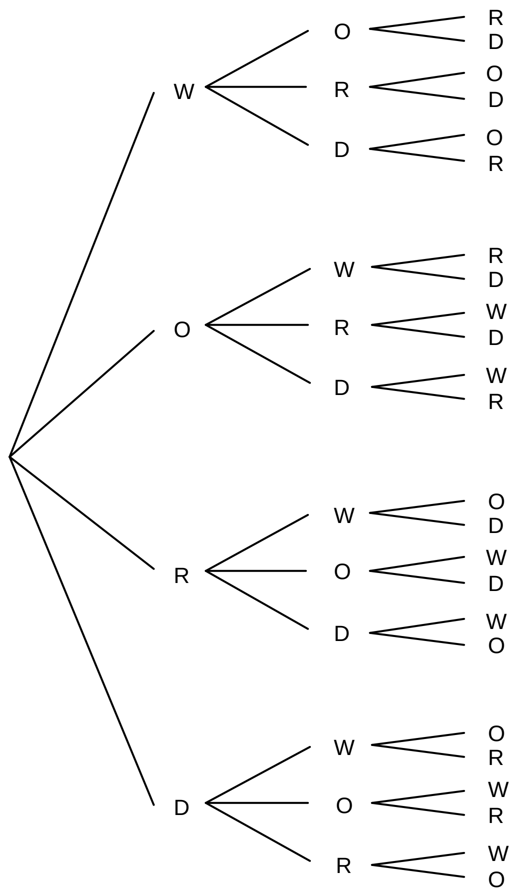
COUNTING, PERMUTATIONS AND COMBINATIONS

MULTIPLICATION PRINCIPLE

Consider the 3 letter words that can be made from the letters WORD if no letter is repeated. These can be listed by means of a tree diagram.



There are:

4 ways of choosing the 1st letter
3 ways of choosing the 2nd letter
2 ways of choosing the 3rd letter
number of words = 4 x 3 x 2 = 24

This is an illustration of the *multiplication principle* ie. if several operations are carried out in a certain order, then the number of ways of performing all the operations is the product of the numbers of ways of performing each operation.

The principle is equivalent to filling in pigeonholes:

1st letter   2nd letter   3rd letter
□   □   □
4 x 3 x 2 = 24

ADDITION PRINCIPLE

Consider the 3 letter words starting <u>or</u> finishing with O that can be made from the letters WORD if no letter is repeated. Now words starting or finishing with O are *mutually exclusive* ie. they do not overlap. Therefore we can find the number starting with O and the number finishing with O and add the two numbers.

<u>starting with O</u>

1st letter   2nd letter   3rd letter
□   □   □
1 x 3 x 2 = 6

<u>finishing with O</u>

1st letter   2nd letter   3rd letter
□   □   □
3 x 2 x 1 = 6

number of words starting or finishing with O = 6 + 6 = 12

This is an illustration of the *addition principle* ie. if two operations are mutually exclusive (ie. they do not overlap), then the number of ways of performing one operation <u>or</u> the other is the sum of the numbers of ways of performing each operation.

FACTORIAL NOTATION

In applying the multiplication principle, factorial notation can be useful eg. the number of 6 letter words that can be made from the letters FACTOR is 6! = 6 x 5 x 4 x 3 x 2 x 1.

"TOGETHER" ARRANGEMENTS

In this type of problem, we need to count arrangements where some of the objects must remain together. The multiplication principle applies and we use a "treat as one" technique.

Eg. 3 science, 4 mathematics and 5 history books are arranged on a shelf. How many arrangements are possible if the books from each subject are to be together?

treat the books for each subject as one book:
number of arrangements = 3!

number of ways of arranging the science books = 3!
number of ways of arranging the mathematics books = 4!
number of ways of arranging the history books = 5!

total number of arrangements = 3! x 3! x 4! x 5! = 103680

ARRANGEMENTS INVOLVING IDENTICAL OBJECTS

Consider the number of arrangements of the letters EMPLOYEE. If all 8 letters were different, then the number of arrangements would be 8! but this number involves counting arrangements more than once. Eg. The 8! arrangements includes 6 versions of EEEMPLOY:

$$E_1 E_2 E_3 MPLOY$$
$$E_1 E_3 E_2 MPLOY$$
$$E_2 E_1 E_3 MPLOY$$
$$E_2 E_3 E_1 MPLOY$$
$$E_3 E_1 E_2 MPLOY$$
$$E_3 E_2 E_1 MPLOY$$

Similarly every arrangement occurs 6 times in the total of 8! (6 is the number of arrangements of the 3 E's ie. 3!).

$$\text{number of distinct arrangements} = \frac{8!}{3!} = 6720$$

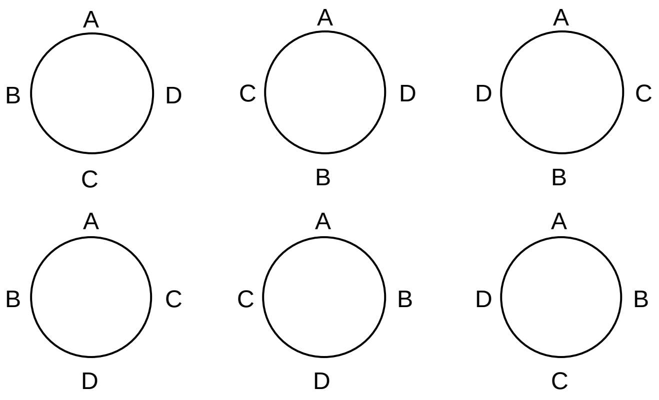
This idea can be extended to problems where more than one type of object is repeated:

$$\text{number of distinct arrangements of the letters MISSISSIPPI} = \frac{11!}{4!4!2!} = 34650$$

CIRCULAR ARRANGEMENTS

Consider the different arrangements when 4 people sit in a circle. There are 3! arrangements (not 4! as might be expected) as it does not matter where the first object is placed.

In general, n objects can be arranged in a circle in $(n-1)!$ ways.



PERMUTATIONS AND COMBINATIONS

Choosing objects from a collection of different objects can be done in several ways. Two particular ways are given the names permutation and combination.

Choosing r objects without repetition from n different objects such that order matters is called a *permutation* and the number of such permutations is denoted by ${}^{n}P_{r}$.

Choosing r objects without repetition from n different objects such that order does not matter is called a *combination* and the number of such combinations is denoted by ${}^{n}C_{r}$.

Example of permutations:
How many ways can a committee of 3 be selected from 7 people A,B,C,D,E,F,G so that there is a president, a vice-president and a secretary? Using the multiplication principle:

pres.		vice-pres.		sec.		
□		□		□		
7	x	6	x	5	=	210

$$\therefore \quad {}^{7}P_{3} = 7 \times 6 \times 5 = \frac{7!}{4!} \quad \text{and in general} \quad {}^{n}P_{r} = \frac{n!}{(n-r)!}$$

NB. Permutation problems are usually best done using the multiplication principle rather than permutation notation.

Example of combinations:
How many ways can a committee of 3 be selected from 7 people A,B,C,D,E,F,G so that each member of the committee is equal? The number of permutations 210 is too large because every combination occurs 6 times in the 210 permutations. Eg. The following 6 permutations each give rise to the same combination (6 is the number of arrangements of 3 objects ie. 3!):

ABC ACB BAC BCA CAB CBA

$$\therefore \quad {}^{7}C_{3} = \frac{7 \times 6 \times 5}{3!} = \frac{7!}{3!4!} \quad \text{and in general} \quad {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

SPECIAL COMBINATIONS

Choosing 7 from 7 so that order does not matter can only be done 1 way:

$$\therefore \quad {}^{7}C_{7} = 1 \quad \text{and in general} \quad {}^{n}C_{n} = 1$$

Choosing 0 from 7 so that order does not matter can only be done 1 way:

$$\therefore \quad {}^{7}C_{0} = 1 \quad \text{and in general} \quad {}^{n}C_{0} = 1$$

This leads to the conclusion that 0! must be given the value 1 because ${}^{7}C_{7} = \frac{7!}{7!0!} = 1$.

COMBINATIONS - INCLUSIONS / EXCLUSIONS

Consider the number of ways a committee of 3 can be selected from 7 people A,B,C,D,E,F,G (order does not matter) if:

B must be included (select 2 from A,C,D,E,F,G) $^6C_2$

D must be excluded (select 3 from A,B,C,E,F,G) $^6C_3$

C and E cannot be chosen together $^5C_3 + {}^5C_2 + {}^5C_2$

What is the justification for the last answer?

CHOOSING SETS OF OBJECTS WITH DISTINCT SUBSETS

In these problems, find the number of ways of choosing each subset and then use the multiplication principle. Eg. 6 people are chosen (order does not matter) from 5 Queenslanders, 4 Tasmanians and 3 Victorians:

2 from each state are chosen:

number of ways of choosing 2 Queenslanders = $^5C_2$

number of ways of choosing 2 Tasmanians = $^4C_2$

number of ways of choosing 2 Victorians = $^3C_2$

total number of ways = $^5C_2 \times {}^4C_2 \times {}^3C_2$

at least 3 Queenslanders are chosen:

number of ways of choosing 3 Queenslanders and 3 others = $^5C_3 \times {}^7C_3$

number of ways of choosing 4 Queenslanders and 2 others = $^5C_4 \times {}^7C_2$

number of ways of choosing 5 Queenslanders and 1 others = $^5C_5 \times {}^7C_1$

total number of ways = $^5C_3 \times {}^7C_3 + {}^5C_4 \times {}^7C_2 + {}^5C_5 \times {}^7C_1$

at least one Queenslander is chosen:

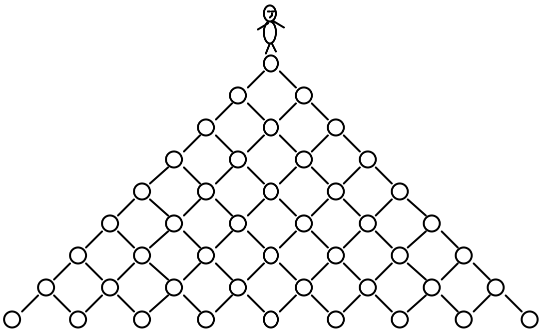
number of ways of choosing without restrictions = $^{12}C_6$

number of ways of choosing with no Queenslanders = $^7C_6$

total number of ways = $^{12}C_6 - {}^7C_6$

PASCAL'S TRIANGLE

Suppose that you start from the top of the triangular arrangement of spaces shown on the attached sheet.



In each space, write the number of shortest possible routes to that space. These numbers form *Pascal's triangle*.

Can you see the pattern which takes you from one row to the next? Justify this pattern in terms of the shortest route problem.

The shortest route problem can be used to explain why each row can be written as combinations eg. the last row is:

$$^8C_8 \quad ^8C_7 \quad ^8C_6 \quad ^8C_5 \quad ^8C_4 \quad ^8C_3 \quad ^8C_2 \quad ^8C_1 \quad ^8C_0$$

Each number in Pascal's triangle (except those on the outside) can be found by adding the pair of numbers immediately above. This recurrence relationship can be written as:

$$^nC_r + {^nC_{r-1}} = {^{n+1}C_r}$$

As well as justifying this recurrence relationship in terms of the shortest route problem, it can be derived in other ways:

- algebraically by writing $^nC_r = \dfrac{n!}{r!(n-r)!}$ etc.
- by considering the selection of r objects from two collections - collection A containing n objects and collection B containing one object.

Consider sloping rows of Pascal's triangle. In turn, they give:

- ones $\{1,1,1,1,1,...\}$
- natural numbers $\{1,2,3,4,5,\ldots\}$
- triangle numbers $\{1,3,6,10,15,\ldots\}$
- tetrahedral numbers $\{1,4,10,20,35,\ldots\}$

What is the connection between natural numbers, triangle numbers and tetrahedral numbers? How do triangle numbers and tetrahedral numbers get their names? What is the nth triangle number and the nth tetrahedral number as a combination? Write these combinations as algebraic expressions in n.

<u>BINOMIAL THEOREM</u>

Consider the *binomial expansion*:

$$(x+y)^8 = (x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)(x+y)$$

The number of x^5y^3 in the expansion is the same as the number of ways of selecting 5 x's from the 8 available ie. $^8C_5$. This leads to the *binomial theorem* for binomial expansions $(x+y)^n$ where n is a positive integer. The theorem gives the coefficients as combinations. Eg. for $n = 8$:

$$(x+y)^8$$
$$= {^8C_8}\,x^8 + {^8C_7}\,x^7y + {^8C_6}\,x^6y^2 + {^8C_5}\,x^5y^3 + {^8C_4}\,x^4y^4 + {^8C_3}\,x^3y^5 + {^8C_2}\,x^2y^6 + {^8C_1}\,xy^7 + {^8C_0}\,y^8$$
$$= x^8 + 8x^7y + 28x^6y^2 + 56x^5y^3 + 70x^4y^4 + 56x^3y^5 + 28x^2y^6 + 8xy^7 + y^8$$

From the earlier work on Pascal's triangle, we can also say that the coefficients of a binomial expansion are given by the appropriate row of Pascal's triangle. Substituting $x = y = 1$ in the binomial theorem shows that the sums of the rows of Pascal's triangle are powers of 2.