DIFFERENTIATION FROM FIRST PRINCIPLES

In previous work, we saw that:

- The instantaneous rate of change of a function is the gradient of the tangent.
- The gradient of the tangent could be estimated numerically as the limit of the gradient of the secant.

gradient of the tangent at the point on the curve where $x = c$

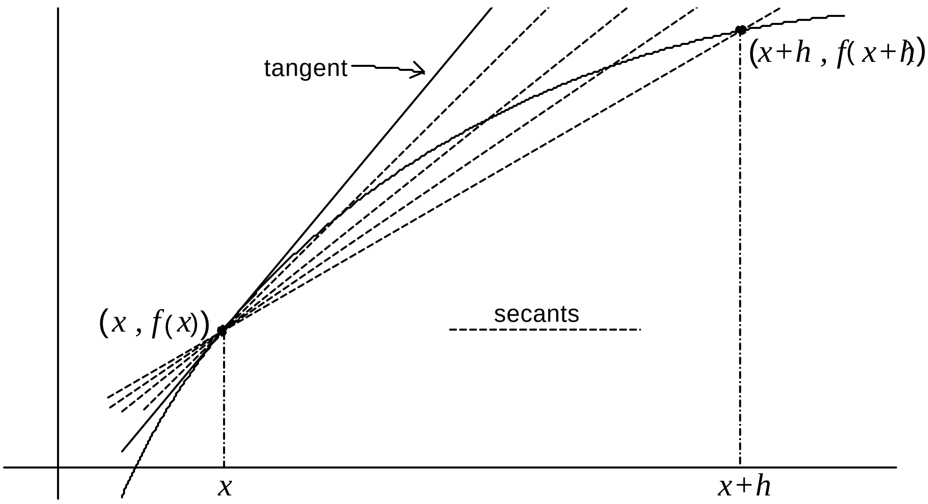
$= \lim_{x \to c} (\text{gradient of secant})$

$= \lim_{x \to c} \dfrac{\Delta f}{\Delta x}$

$= \lim_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

Differentiation from first principles is a method for finding the gradient of the tangent by using the limit process in a more formal way. Most importantly, it allows us to find the gradient of the tangent at any point.

Because we are considering any point, we will take the point to be $(x, f(x))$.
Also it is easier to represent Δx by h.



The function $f'(x)$ is used to denote the gradient of the tangent at $(x, f(x))$.

$f'(x)$ is called the *derivative* or *gradient function*.

Where the limit exists:

$$f'(x) = \lim_{\Delta x \to 0} (\text{gradient of secant}) = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Examples of differentiation from first principles are shown below.

Finding the derivative of $f(x) = 4x^2 - 5x$	
at the point $(2,6)$	at any point
$f'(2)$ $= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h}$ $= \lim_{h \to 0} \frac{\left[4(2+h)^2 - 5(2+h)\right] - \left[4 \times 2^2 - 5 \times 2\right]}{h}$ $= \lim_{h \to 0} \frac{\left[4\left(4 + 4h + h^2\right) - 5(2+h)\right] - 6}{h}$ $= \lim_{h \to 0} \frac{16 + 16h + 4h^2 - 10 - 5h - 6}{h}$ $= \lim_{h \to 0} \frac{11h + 4h^2}{h}$ $= \lim_{h \to 0} \frac{h(11 + 4h)}{h}$ $= \lim_{h \to 0} (11 + 4h)$ $= 11$	$f'(x)$ $= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $= \lim_{h \to 0} \frac{\left[4(x+h)^2 - 5(x+h)\right] - \left[4x^2 - 5x\right]}{h}$ $= \lim_{h \to 0} \frac{\left[4\left(x^2 + 2xh + h^2\right) - 5(x+h)\right] - \left[4x^2 - 5x\right]}{h}$ $= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 5x - 5h - 4x^2 + 5x}{h}$ $= \lim_{h \to 0} \frac{8xh - 5h + 4h^2}{h}$ $= \lim_{h \to 0} \frac{h(8x - 5 + 4h)}{h}$ $= \lim_{h \to 0} (8x - 5 + 4h)$ $= 8x - 5$