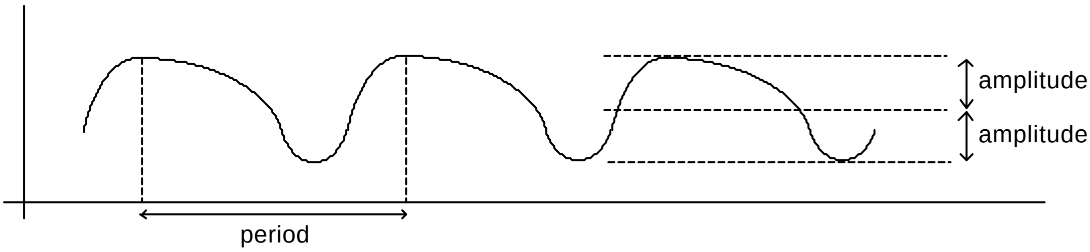
PERIODIC FUNCTIONS & APPLICATIONS

PERIODIC FUNCTIONS

- *Periodic functions* have values which occur in a repeating *cycle* or *pattern*. The length of the cycle is known as the *period*.
- The *frequency* is the number of cycles occurring in a unit of time:

$$\text{frequency} = \frac{1}{\text{period}}$$

  If the period is measured in hours, then the frequency is measured in cycles per hour, etc. 1 hertz (Hz) is 1 cycle per second.
- The *amplitude* is the distance of each peak or trough from the mean value. Where this distance varies, the *average amplitude* can be calculated by halving the average peak to trough distance (see example on page 445).



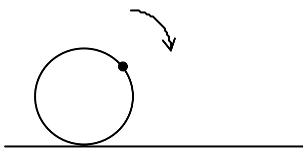
☺ Graph each of the following functions and state its period. $[x]$ is defined as the largest integer that is less than or equal to x.

$$y = x - [x] \qquad y = (x - [x])^2 \qquad y = \frac{x}{2} - \left[\frac{x}{2}\right]$$

☺ Write down the first 10 terms of each of the following discrete recursive functions and state its period.

$$T_1 = 2, \quad T_2 = 3, \quad T_n = T_{n-1} - T_{n-2} \quad (n \geq 3)$$
$$T_1 = 3, \quad T_2 = 2, \quad T_3 = 1, \quad T_n = T_{n-1} - T_{n-2} + T_{n-3} \quad (n \geq 4)$$
$$T_1 = 1, \quad T_2 = 2, \quad T_3 = 1, \quad T_n = T_{n-1} + T_{n-2} - T_{n-3} \quad (n \geq 4)$$
$$T_1 = 1, \quad T_n = T_{n-1} \times (-1)^n \quad (n \geq 2)$$
$$T_1 = 3, \quad T_n = \frac{1}{T_{n-1}} \quad (n \geq 2)$$

☺ Cut out a circle and mark a point on its circumference. Roll the circle along a straight line and mark the path traced out by the point. This curve is called a *cycloid*. What is the period? If the graph is considered as a function, what do the domain and range represent?

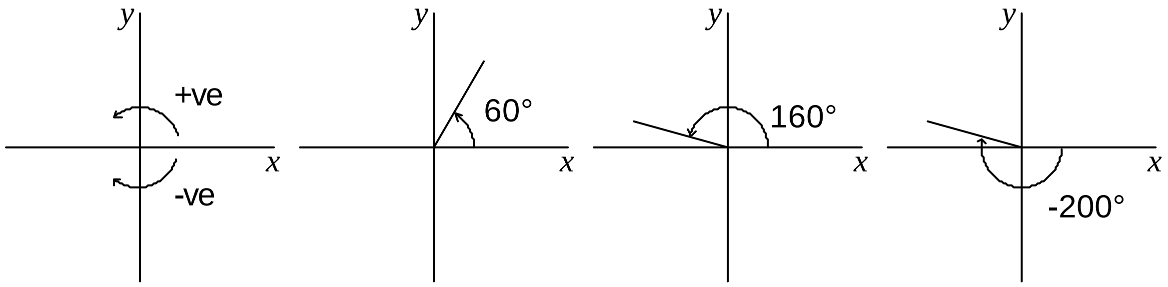


☺ Ex 12.1 p 446

TRIGONOMETRIC RATIOS - ANY ANGLES

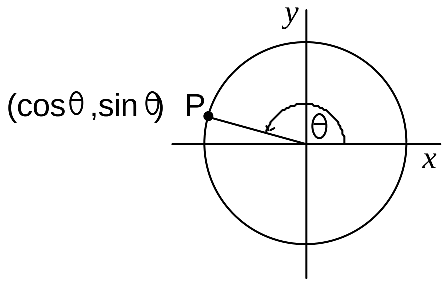
In a previous topic, we defined trigonometric ratios for any angle

On the Cartesian plane, angles are measured *from the positive x-axis* with *anticlockwise* taken as *positive* as shown below. NB. The angles -200° and 160° are represented by the same line segment.



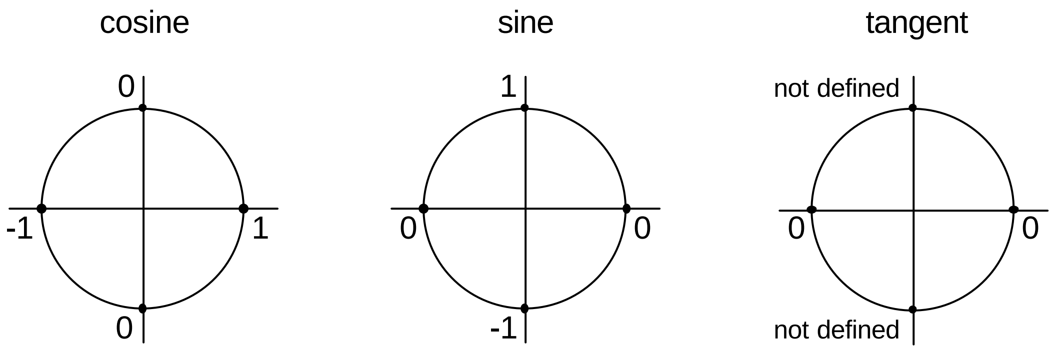
To define trigonometric ratios for any angle θ:

- Draw the *unit circle* (radius 1 and centre the origin).
- Draw the angle.
- Consider the point P on the unit circle.
- $\cos\theta$ is the x-coordinate of P.
- $\sin\theta$ is the y-coordinate of P.
- $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$



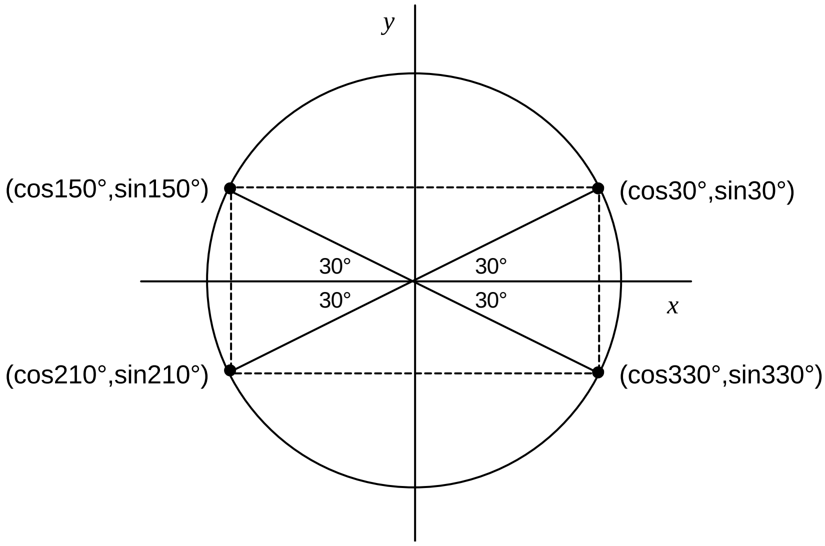
TRIGONOMETRIC RATIOS - MULTIPLES OF 90° OR π/2 RADIANS

Using the definitions gives:



TRIGONOMETRIC RATIOS - ANGLES OTHER THAN MULTIPLES OF 90° OR π/2 RADIANS

Consider the angles whose terminal sides make an angle of 30° with the x-axis (we will call this the *associated acute angle*). 30°, 150°, 210° and 330° each have an associated acute angle of 30°. Consider the points on the unit circle.



comparing x-coordinates	$\cos 150^\circ = {}^-\cos 30^\circ$	$\cos 210^\circ = {}^-\cos 30^\circ$	$\cos 330^\circ = \cos 30^\circ$
comparing y-coordinates	$\sin 150^\circ = \sin 30^\circ$	$\sin 210^\circ = {}^-\sin 30^\circ$	$\sin 330^\circ = {}^-\sin 30^\circ$
using the definition of tangent	$\tan 150^\circ$ $= \frac{\sin 150^\circ}{\cos 150^\circ}$ $= \frac{\sin 30^\circ}{{}^-\cos 30^\circ}$ $= -\frac{\sin 30^\circ}{\cos 30^\circ}$ $= {}^-\tan 30^\circ$	$\tan 210^\circ$ $= \frac{\sin 210^\circ}{\cos 210^\circ}$ $= \frac{{}^-\sin 30^\circ}{{}^-\cos 30^\circ}$ $= \frac{\sin 30^\circ}{\cos 30^\circ}$ $= \tan 30^\circ$	$\tan 330^\circ$ $= \frac{\sin 330^\circ}{\cos 330^\circ}$ $= \frac{{}^-\sin 30^\circ}{\cos 30^\circ}$ $= -\frac{\sin 30^\circ}{\cos 30^\circ}$ $= {}^-\tan 30^\circ$

These results suggest the following method for finding the trigonometric ratio of an angle which is not a multiple of 90° or $\pi/2$ radians:

- find the same trigonometric ratio of the associated acute angle
- choose positive/negative according to the quadrant (ASTC gives the positive ratio)

Sine	All
Tangent	Cosine

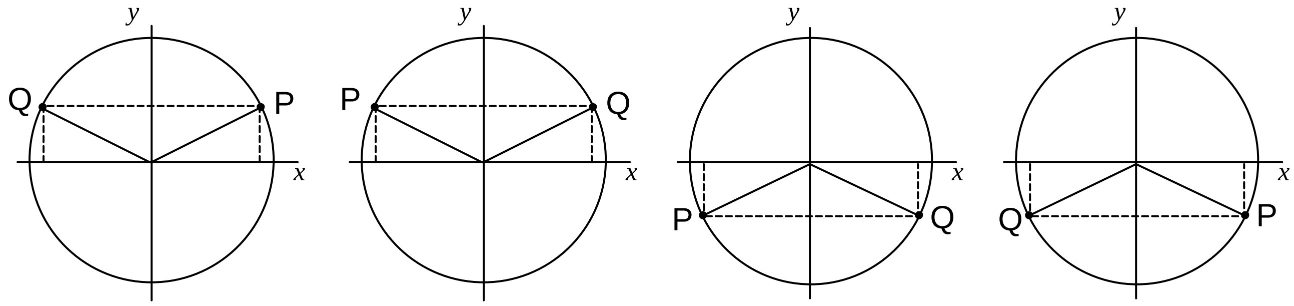
SIMPLIFYING TRIGONOMETRIC RATIOS

Some trigonometric ratios can be simplified. In what follows q can be in any quadrant and $\cot q$ is the reciprocal ratio cotangent ($1/\tan q$).

supplementary angles	$\cos(p - q) = {}^-\cos q$	$\sin(p - q) = \sin q$	$\tan(p - q) = {}^-\tan q$
complementary angles	$\cos\left(\frac{p}{2} - q\right) = \sin q$	$\sin\left(\frac{p}{2} - q\right) = \cos q$	$\tan\left(\frac{p}{2} - q\right) = \cot q$
negative angles	$\cos({}^-q) = \cos q$	$\sin({}^-q) = {}^-\sin q$	$\tan({}^-q) = {}^-\tan q$

The above results can be quickly established by considering points on the unit circle.

For supplementary angles, suppose P is the point on the unit circle representing angle q and Q is the point representing angle $p - q$. P is $(\cos q, \sin q)$ and Q is $(\cos(p - q), \sin(p - q))$.



The following applies regardless of the quadrant of P.

comparing x-coordinates of P and Q: $\cos(p - q) = {}^{-}\cos q$

comparing y-coordinates of P and Q: $\sin(p - q) = \sin q$

using the definition of tangent: $\tan(p - q) = \dfrac{\sin(p - q)}{\cos(p - q)} = \dfrac{\sin q}{{}^{-}\cos q} = -\dfrac{\sin q}{\cos q} = {}^{-}\tan q$

☺ Use the above method to simplify the trigonometric ratios for:

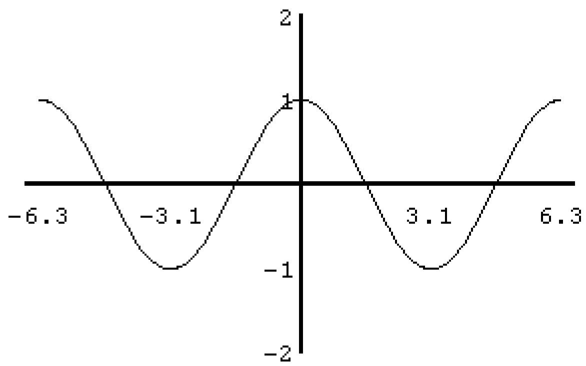
$$\tfrac{p}{2} - q \quad , \quad -q \quad , \quad p + q \quad , \quad 2p - q \quad , \quad \tfrac{p}{2} + q \quad , \quad \tfrac{3p}{2} - q \quad , \quad \tfrac{3p}{2} + q$$

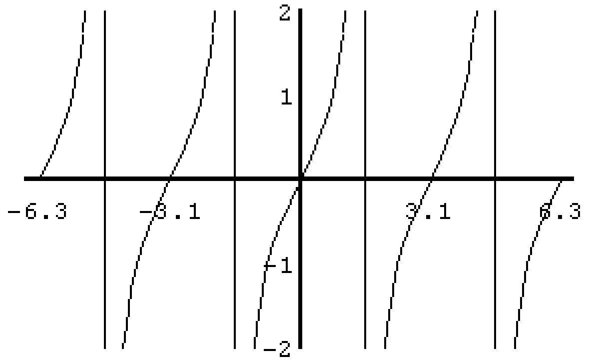
☺ Ex 12.2 p 454 without calculator qu 1,2,6,7,8,13

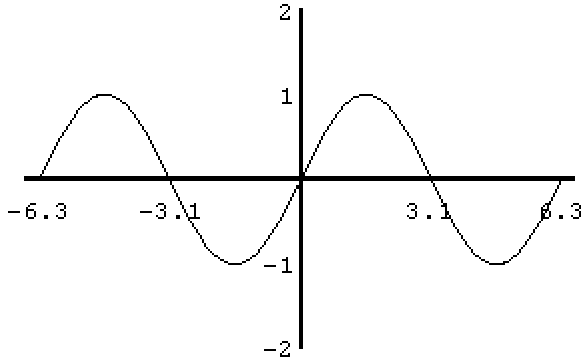
GRAPHS OF COSINE, SINE & TANGENT FUNCTIONS

The graphs of the periodic cosine, sine and tangent functions should be memorised.

The period of the cosine and sine functions is $2p$ and the period of the tangent function is p.







☺ Describe the patterns in the graphs of the cosine, sine and tangent functions.

☺ What is the connection between the graphs of the cosine and sine functions?

☺ Use the graphs to explain some of the earlier results for simplifying trigonometric ratios.

GRAPHS OF OTHER TRIGONOMETRIC FUNCTIONS

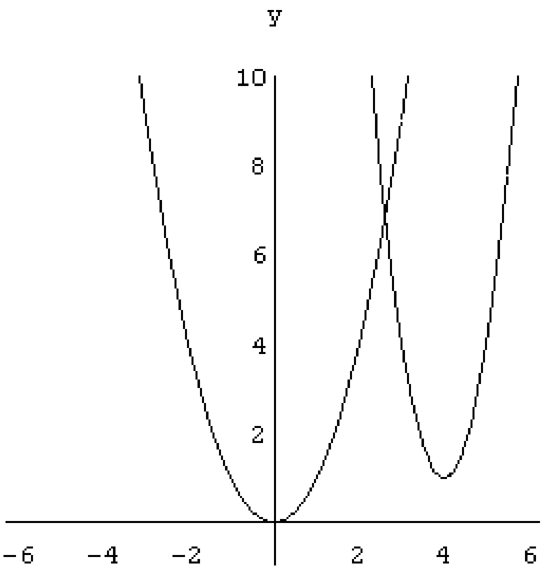
In year 10 and later in topic 5, we sketch graphs using transformations of the graph of $y = x^2$ ie. translations parallel to an axis and dilations parallel to an axis.

$y = f(x)$ graph
dilated by factor a parallel to y-axis
translated h units $\rightarrow$ parallel to x-axis
translated k units $\uparrow$ parallel to y-axis
gives

$\dfrac{y-k}{a} = f(x-h)$ or $y = a\,f(x-h)+k$ graph

$y = x^2$ graph
dilated by factor 3 parallel to y-axis
translated 4 units $\rightarrow$ parallel to x-axis
translated 1 unit $\uparrow$ parallel to y-axis
gives

$\dfrac{y-1}{3} = (x-4)^2$ or $y = 3(x-4)^2+1$ graph



This can be extended to include dilations parallel to both axes:

$y = f(x)$ graph
dilated by factor a_1 parallel to x-axis
dilated by factor a_2 parallel to y-axis
translated h units $\rightarrow$ parallel to x-axis
translated k units $\uparrow$ parallel to y-axis
gives

$\dfrac{y-k}{a_2} = f\left(\dfrac{x-h}{a_1}\right)$ or $y = a_2\, f\left(\dfrac{x-h}{a_1}\right)+k$ graph

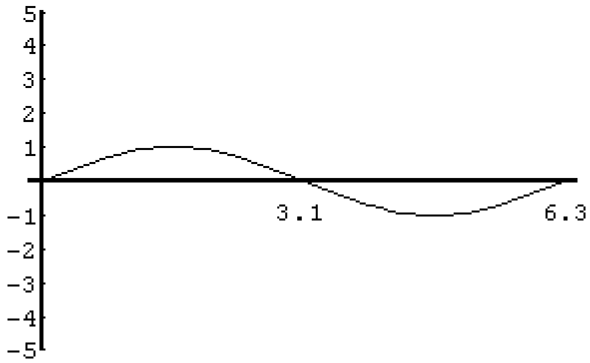
Eg. Sketch the graph of $y = 3\sin(2q - p) + 2$. Rearranging the equation:

$$y = 3\sin(2q - p) + 2$$
$$y - 2 = 3\sin(2q - p)$$
$$\frac{y-2}{3} = \sin(2q - p)$$
$$\frac{y-2}{3} = \sin 2\left(q - \tfrac{p}{2}\right)$$
$$\frac{y-2}{3} = \sin\left(\frac{q - \frac{p}{2}}{\frac{1}{2}}\right)$$

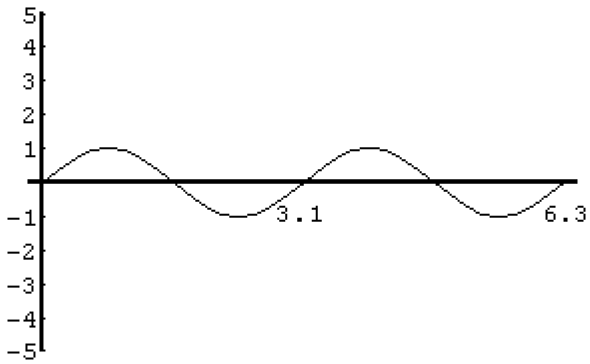
NB. For the trigonometric functions $\dfrac{y-k}{a_2} = \cos\left(\dfrac{q-h}{a_1}\right)$ and $\dfrac{y-k}{a_2} = \sin\left(\dfrac{q-h}{a_1}\right)$:

- the translation parallel to the q-axis is called the *phase change* or *phase shift*
- phase angle = h
- period = $a_1 \times 2p$
- amplitude = a_2

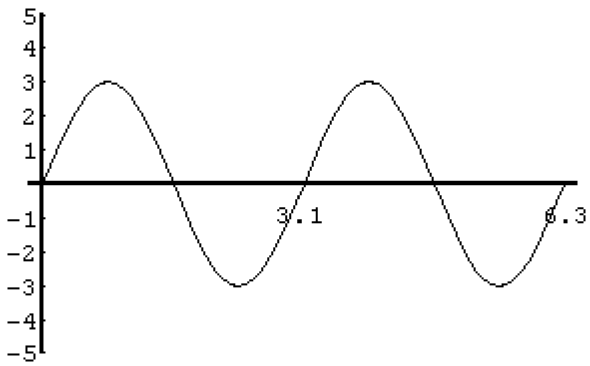
graph of $y = \sin\theta$



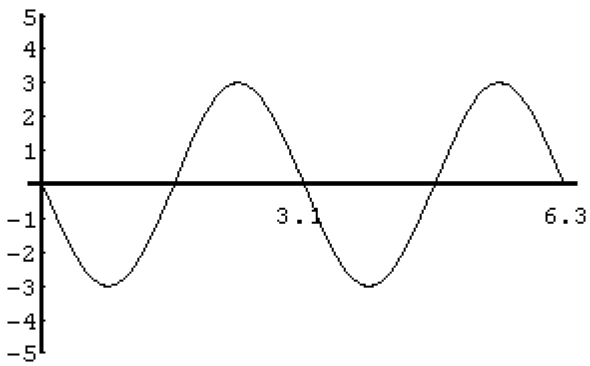
dilate by factor $\frac{1}{2}$ parallel to θ-axis
period = $\frac{1}{2} \times 2\pi = \pi$



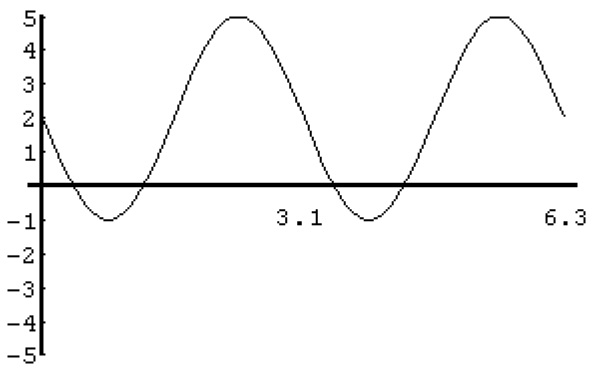
dilate by factor 3 parallel to y-axis
amplitude = 3



translate $\frac{\pi}{2}$ units $\rightarrow$ parallel to θ-axis



translate 2 units $\uparrow$ parallel to y-axis



☺ Ex 12.3 p 464 Ex 12.4 p 473

☺ Use a graphics calculator to investigate (a) the sum of two periodic functions with the same period, (b) the sum of two periodic functions with different periods.