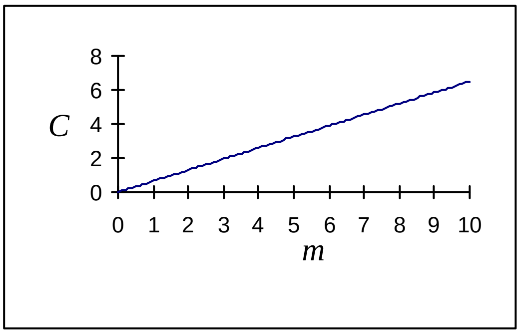
RATES OF CHANGE

A *rate* is used to compare quantities of different types.

If 4·2 kg of potatoes cost \$2·73, then the rate is 0·65 \$/kg.

$$\text{rate} = \frac{\text{quantity 2}}{\text{quantity 1}} = \frac{\$2\cdot 73}{4\cdot 2\text{ kg}} = 0\cdot 65\ \$/\text{kg}$$

This is an example of a constant rate. If a graph of one quantity against the other is plotted, then a straight line through the origin is obtained. The rate 0·65 \$/kg is the gradient of the line and the equation of the graph is $C = 0\cdot 65m$ where \$$C$ is the cost of m kg.



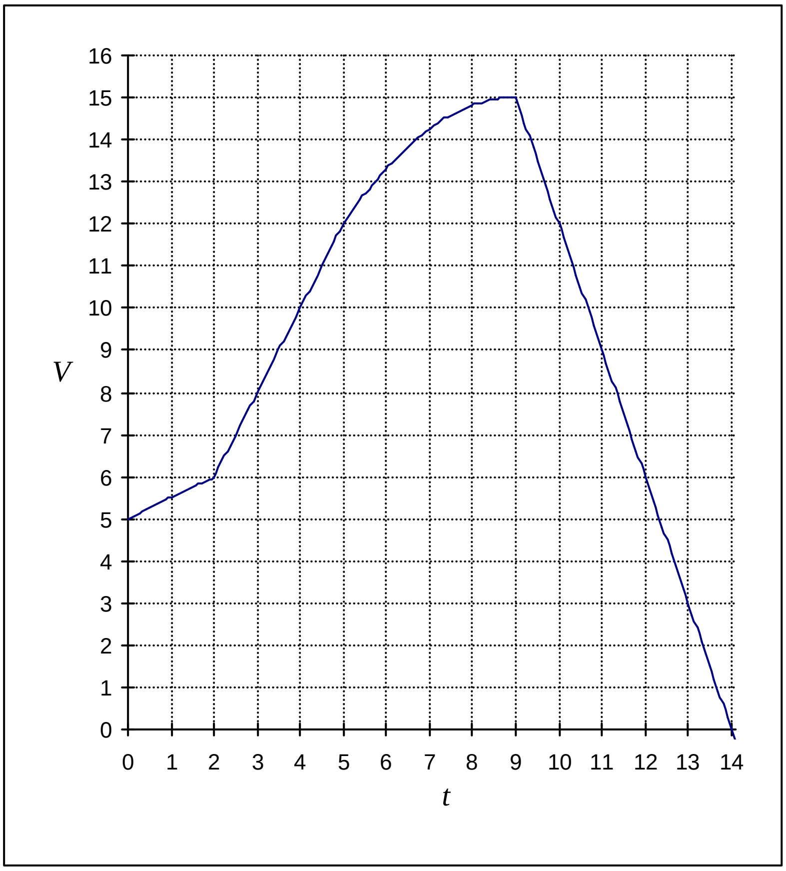
Instead of comparing the quantities themselves, we may wish to compare changes in those quantities. This involves extending the idea of *rate* to that of *rate of change*.

Consider a water tank with several inlets and outlets. Suppose the volume of water in the tank V kL is plotted against time t hours as shown in the graph opposite.

In what way did the volume change with respect to time (ie. how did changes in volume compare with changes in time):

- between $t = 0$ and $t = 2$
- between $t = 2$ and $t = 5$
- between $t = 5$ and $t = 9$
- between $t = 9$ and $t = 14$?

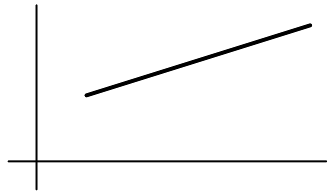
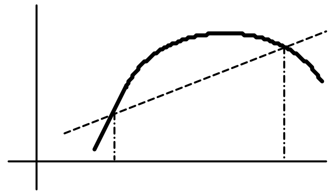
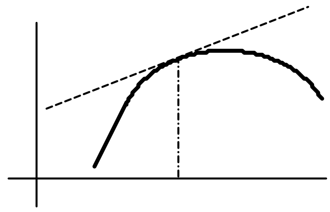
To answer this question, we have to talk about the steepness or the gradient of the graph. This leads to the definition of *rate of change*.



The rate of change of quantity 2 with respect to quantity 1 is defined as the gradient of the graph obtained with quantity 2 on the vertical axis and quantity 1 on the horizontal axis.

Remember the formula for the gradient of the straight line joining two points.

$$\text{gradient} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

There are 3 different types of rates of change.		
constant rate of change	the gradient of the *line*	
average rate of change	the gradient of the *secant*	
instantaneous rate of change	the gradient of the *tangent*	

Examples from the tank problem.
constant rate of change of volume with respect to time between $t = 2$ and $t = 5$ $= \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{12 - 6}{5 - 2} = 2$ kL / h
constant rate of change of volume with respect to time between $t = 9$ and $t = 14$ $= \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{0 - 15}{14 - 9} = -3$ kL / h
average rate of change of volume with respect to time between $t = 2$ and $t = 9$ $= \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{15 - 6}{9 - 2} = 1 \cdot 3$ kL / h
average rate of change of volume with respect to time between $t = 5$ and $t = 11$ $= \dfrac{V_2 - V_1}{t_2 - t_1} = \dfrac{9 - 12}{11 - 5} = -0 \cdot 5$ kL / h
instantaneous rate of change at $t = 7$ $= \frac{3}{4}$ kL / h (by drawing the tangent)

SPECIAL WORDS AND IDEAS

speed	rate of change of distance with respect to time (gradient of distance-time graph)
average speed	gradient of a secant on the distance-time graph or $\frac{\text{total distance travelled}}{\text{total time taken}}$
instantaneous speed	gradient of a tangent on the distance-time graph

displacement	distance travelled in a certain direction

velocity	rate of change of displacement with respect to time (gradient of displacement-time graph)
instantaneous velocity	gradient of a tangent on the displacement -time graph
average velocity	gradient of a secant on the displacement-time graph $\frac{\Delta s}{\Delta t} = \frac{s_2 - s_1}{t_2 - t_1} = \frac{\text{final displacement - initial displacement}}{\text{total time taken}}$

acceleration	rate of change of velocity with respect to time (gradient of the velocity-time graph)

vector	quantity with direction as well as magnitude • Displacement, velocity and acceleration are vectors. • Distance and speed are not vectors. • “Distance” is often used loosely instead of “displacement”.

FUNCTION NOTATION

average rate of change of $f(x)$ with respect to x
between $x = x_1$ and $x = x_2$

$=$ gradient of secant

$$= \frac{\Delta f}{\Delta x}$$

$$= \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$