11 MATHEMATICS C - REVIEW - GROUPS

1) What is the name given to a set of elements that obeys the closure, commutative, associative, identity, inverse and distributive properties under addition and multiplication?
(This is a question for those students who have studied the first page of the notes on real numbers.)

2) Invent four examples of a binary operation on the real numbers. Your answers should not include addition, subtraction, multiplication, division or any of the operations mentioned in this review sheet.

3) What is the name given to a set of elements that obeys the closure, associative, identity and inverse properties under a binary operation?

4) What special name is given to a group for which the binary operation is commutative?

5) What is true for all groups with 4 or fewer elements?

6) A Cayley table shows the results of a binary operation on a finite set of elements and contains a row with an element repeated. What does this tell you?

7) Two binary operations $\circ$ and $*$ are defined on the real numbers:

$$a \circ b = 2\left(a+b\right)$$

$$a * b = 2\,a\,b$$

Show that only one of the operations is associative.

8) Show that each of the following does not form a group:
   a) $\left\{\frac{1}{125}\ ,\ \frac{1}{25}\ ,\ \frac{1}{5}\ ,\ 1\ ,\ 5\ ,\ 25\ ,\ 125\right\}$ under multiplication
   b) $\{1,2,3,4,5\}$ under multiplication mod 6
   c) $\{1,2,3,4,5\}$ under addition mod 6
   d) the set of rational numbers under subtraction
   e) the integers under the operation $a \circ b = \left| a+b \right|$

9) Show that the set of integers $\{0,3,6,9,12\}$ forms a group under addition modulo 15.
You may assume that the associative property holds.

10) The real numbers form a group under the operation $a \circ b = a+b+7$.
   a) Explain why $^{-}7$ is the identity element?
   b) Given any element a, what is its inverse?

11) Complete a Cayley tale for the group consisting of the integers $\{1,2,3,4,5,6\}$ under multiplication mod 7. Find three subgroups other than $\{1,2,3,4,5,6\}$.

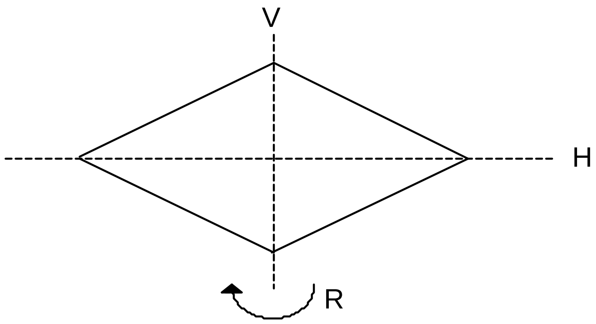
12) The Cayley table below represents a group. Show that the group is isomorphic to the group consisting of the integers $\{1,2,3,4,5,6\}$ under multiplication mod 7.

$\circ$	a	b	c	d	e	f
a	e	c	a	b	f	d
b	c	d	b	f	a	e
c	a	b	c	d	e	f
d	b	f	d	e	c	a
e	f	a	e	c	d	b
f	d	e	f	a	b	c

13) Construct a Cayley table for set of eight integers that form a group under multiplication modulo 30. (Do not prove that the set is a group.) Find all the possible subgroups.

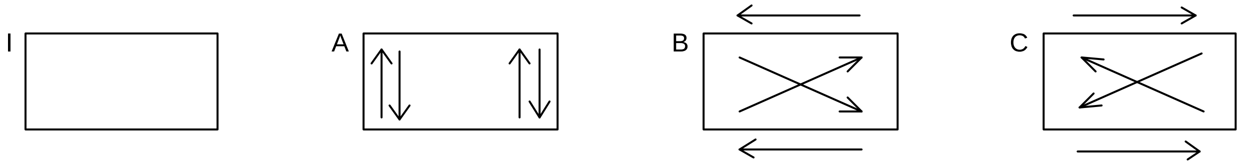
14) Complete the table below for the symmetries of a rhombus. The binary operation $\circ$ is "followed by" (e.g. H $\circ$ V means V followed by H).

I is the identity transformation
H is a reflection in the horizontal axis
V is a reflection in the vertical axis
R is a rotation of 180° about the centre



$\circ$	I	H	V	R
I				
H				
V				
R				

15) Below is a set of four possible plans for changing the tyres of a car to reduce wear. Complete a Cayley table and show that the set forms a group under the operation "followed by" (e.g. A $\circ$ B means B followed by A). You can assume that the associative property holds.



16) The set $\{e, a, b\}$ forms a group and e is the identity. Show that there is only one possible Cayley table.

(7) $5\circ(3\circ1)$ $(5\circ3)\circ1$

$= 5\circ8$ $= 16\circ1$

$= 26$ $= 34$

$5\circ(3\circ1) \neq (5\circ3)\circ1$

$\therefore$ The operation is not associative.

Consider any real numbers a, b, c:

$a*(b*c)$ $(a*b)*c$

$= a*2bc$ $= (2ab)*c$

$= 2(a)(2bc)$ $= 2(2ab)c$

$= 4abc$ $= 4abc$

$a*(b*c) = (a*b)*c$

$\therefore$ The operation is associative.

(8)(a) $\{\frac{1}{125}, \frac{1}{25}, \frac{1}{5}, 1, 5, 25, 125\}$ under multiplication

$5\times125 = 625$ which is not in the set

$\therefore$ The closure property does not hold.

$\therefore$ The set is not a group.

(b) $\{1, 2, 3, 4, 5\}$ under multiplication mod 6

$2\times3 = 0$ which is not in the set

$\therefore$ The closure property does not hold.

$\therefore$ The set is not a group.

(c) $\{1, 2, 3, 4, 5\}$ under addition mod 6

$2+4 = 0$ which is not in the set

$\therefore$ The closure property does not hold.

$\therefore$ The set is not a group.

(d) the set of rational numbers under subtraction

$$5-(4-1) \qquad (5-4)-1$$
$$= 5-3 \qquad = 1-1$$
$$= 2 \qquad = 0$$

$$5-(4-1) \neq (5-4)-1$$

∴ The associative property does not hold.

∴ The set is not a group.

(e) the integers under the operation $a \circ b = |a+b|$

$$3 \circ (-7 \circ 5) \qquad (3 \circ {}^{-}7) \circ 5$$
$$= 3 \circ |{}^{-}7+5| \qquad = |3 + {}^{-}7| \circ 5$$
$$= 3 \circ |{}^{-}2| \qquad = |{}^{-}4| \circ 5$$
$$= 3 \circ 2 \qquad = 4 \circ 5$$
$$= |3+2| \qquad = |4+5|$$
$$= |5| \qquad = 9$$
$$= 5$$

$$3 \circ (-7 \circ 5) \neq (3 \circ {}^{-}7) \circ 5$$

∴ The associative property does not hold.

∴ The set is not a group.

(a) Consider the Cayley table:

+	0	3	6	9	12
0	0	3	6	9	12
3	3	6	9	12	0
6	6	9	12	0	3
9	9	12	0	3	6
12	12	0	3	6	9

Each result in the table is an element of the set.

∴ The closure property applies.

The Cayley table shows that 0 is the identity.

$\therefore$ The identity property applies.

$3+12 = 12+3 = 0$

$\therefore$ 3 and 12 are inverses

$6+9 = 9+6$

$\therefore$ 6 and 9 are inverses

$\therefore$ Each element has an inverse.

$\therefore$ The inverse property applies.

$\therefore$ The set forms a group because the closure, associative, identity and inverse properties apply.

(10) (a) Consider any real number a.

$$\begin{aligned} & a \circ {}^{-}7 \\ &= a + {}^{-}7 + 7 \\ &= a - 7 + 7 \\ &= a \end{aligned} \qquad\qquad \begin{aligned} & {}^{-}7 \circ a \\ &= {}^{-}7 + a + 7 \\ &= a + 7 - 7 \\ &= a \end{aligned}$$

$\therefore \; a \circ {}^{-}7 = {}^{-}7 \circ a = a$

$\therefore \; {}^{-}7$ is the identity

(b) Let the inverse of a be b.

$$\begin{aligned} a \circ b &= {}^{-}7 \\ a + b + 7 &= {}^{-}7 \\ b &= {}^{-}14 - a \end{aligned}$$

but $b \circ a = ({}^{-}14 - a) + a + 7 = {}^{-}14 - a + a + 7 = {}^{-}7$

$\therefore \; {}^{-}14 - a$ is the inverse of a.

×	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

multiplication mod 7

The identity is 1.
2 and 4 are inverses.
3 and 5 are inverses.
6 is its own inverse.

The order of a subgroup must be 1, 2 or 3 (a factor of 6 by Lagrange's Theorem).

Any element in a subgroup must be accompanied by its inverse.

Subgroups:

$\{1\}$

$\{1, 6\}$

$\{1, 2, 4\}$

(12) The Cayley table shows that, combining c with any element, gives the identical element. Therefore c is the identity.

$\therefore$ c matches 1

$f \circ f = c$

$\therefore$ f is its own universe

$\therefore$ f matches 6

$a \circ b = b \circ a = c$ and $d \circ e = e \circ d = c$

$\therefore$ a and b match 3 and 5

or a and b match 2 and 4

$a \circ a = e$ and $2 \times 2 = 4$

This shows that a and b do not match 2 and 4.

In the Cayley table, replace c with 1, f with 6, a with 3, b with 5, d with 2 and e with 4:

$\circ$	3	5	1	2	4	6
3	4	1	3	5	6	2
5	1	2	5	6	3	4
1	3	5	1	2	4	6
2	5	6	2	4	1	3
4	6	3	4	1	2	5
6	2	4	6	3	5	1

This does not work as $3 \times 3 \neq 4$.

But $3 \times 3 = 2$, therefore replace d with 4 and e with 2.

The resulting Cayley table can also be produced by multiplication modulo 7. Therefore the two groups are isomorphic.

o	3	5	1	4	2	6
3	2	1	3	5	6	4
5	1	4	5	6	3	2
1	3	5	1	4	2	6
4	5	6	4	2	1	3
2	6	3	2	1	4	5
6	4	2	6	3	5	1

(13) Consider the integers that do not have a common factor (other than 1) with 30:

$\{1, 7, 11, 13, 17, 19, 23, 29\}$

The identity is 1.
11 is its own inverse.
19 is its own inverse.
29 is its own inverse.
7 and 13 are inverses.
17 and 23 are inverses.

x	1	7	11	13	17	19	23	29
1	1	7	11	13	17	19	23	29
7	7	19	17	1	29	13	11	23
11	11	17	1	23	7	29	13	19
13	13	1	23	19	11	7	29	17
17	17	29	7	11	19	23	1	13
19	19	13	29	7	23	1	17	11
23	23	11	13	29	1	17	19	7
29	29	23	19	17	13	11	7	1

The order of a subgroup must be 1, 2 or 4 (a factor of 8 by Lagrange's Theorem).

The subgroups are:

$\{1\}$

$\{1, 11\}$, $\{1, 19\}$, $\{1, 29\}$

$\{1, 11, 19, 29\}$, $\{1, 19, 7, 13\}$, $\{1, 29, 17, 23\}$

NB. If an element is in the subgroup, then so is its inverse.

(14)

$\circ$	I	H	V	R
I	I	H	V	R
H	H	I	R	V
V	V	R	I	H
R	R	V	H	I

(15)

$\circ$	I	A	B	C
I	I	A	B	C
A	A	I	C	B
B	B	C	A	I
C	C	B	I	A

I is the identity.
Therefore the identity property applies.

$A \circ A = I$ and $B \circ C = C \circ B = I$
Therefore the inverse property applies.

Each result in the table is in the set $\{I, A, B, C\}$
Therefore the closure property applies.

Therefore the set forms a group as the closure, associative, identity and inverse properties apply.

(16) Consider the Cayley table.

$\circ$	e	a	b
e	e	a	b
a	a	b	e
b	b	e	a

The first row and first column are completed using the identity property.

Elements are not repeated in a row.

$\therefore$ $a \circ a = e$ and $a \circ b = b$

or $a \circ a = b$ and $a \circ b = e$

The first option is not possible as b would be repeated in the third column.

$\therefore$ $a \circ a = b$ and $a \circ b = e$.

Ensuring no repetitions in a row or column, the table can only be completed as follows:

$b \circ a = e$ and $b \circ b = a$