

MATHEMATICS C  
LINEAR PROGRAMMING  
SIMPLEX METHOD

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## SIMPLEX METHOD - STANDARD MAXIMISATION PROBLEM

*Standard maximisation problem - a linear programming problem for which the objective function is to be maximised and all the constraints are "less-than-or-equal-to" inequalities.*

The Cannon Hill Furniture Company produces chairs and tables. Each table takes four hours of labour from the carpentry department and two hours of labour from the finishing department. Each chair requires three hours of carpentry and one hour of finishing. During the current week, 240 hours of carpentry time are available and 100 hours of finishing time. Each table produced gives a profit of \$70 and each chair a profit of \$50. How many chairs and tables should be made?

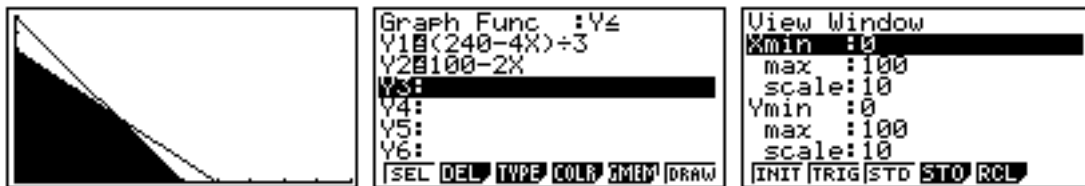
We first choose the variables - suppose  $x$  tables and  $y$  chairs are produced in the week.

The information can be summarised:

	Tables	Chairs	Constraints	
Number produced per week	$x$	$y$	Cannot be negative	$x \geq 0, y \geq 0$
carpentry	4 h/table	3 h/chair	Maximum of 240 hours for the week	$4x + 3y \leq 240$
finishing	2 h/table	1 h/chair	Maximum of 100 hours for the week	$2x + y \leq 100$
Profit	\$70 per table	\$50 per chair		

The total profit (\$) for the week is given by the objective function  $P = 70x + 50y$ .

As decisions need to be made about the values of two variables  $x$  and  $y$ , the possible values can be represented as a shaded region on a two-dimensional diagram. From previous work, we know that, if the problem has a solution, then the solution occurs at one of the vertices of the shaded area.



When the simplex method is used in the furniture problem, the objective function is written in terms of four variables. If the problem has a solution, then the solution occurs at one of the vertices of a region in four-dimensional space. We start at one of the vertices and check the neighbouring vertices to see which ones provide a better solution. We then move to one of the vertices that give a better solution. The process is repeated until the target vertex is reached.

The first step of the simplex method requires that each inequality be converted into an equation. "Less-than-or-equal-to" inequalities are converted to equations by including slack variables. Suppose  $s_1$  carpentry hours and  $s_2$  finishing hours remain unused in a week. The constraints become:

$$\begin{aligned} 4x + 3y + s_1 &= 240 & \text{or} & & 4x + 3y + 1s_1 + 0s_2 &= 240 \\ 2x + y + s_2 &= 100 & & & 2x + 1y + 0s_1 + 1s_2 &= 100 \end{aligned}$$

As unused hours result in no profit, the slack variables can be included in the objective function with zero coefficients:

$$\begin{aligned} P &= 70x + 50y + 0s_1 + 0s_2 \\ -70x - 50y + 0s_1 + 0s_2 + 1P &= 0 \end{aligned}$$

The problem can now be considered as solving a system of 3 linear equations involving the 5 variables  $x, y, s_1, s_2, P$  in such a way that  $P$  has the maximum value:

$$\begin{aligned} 4x + 3y + 1s_1 + 0s_2 + 0P &= 240 \\ 2x + 1y + 0s_1 + 1s_2 + 0P &= 100 \\ -70x - 50y + 0s_1 + 0s_2 + 1P &= 0 \end{aligned}$$

The system of linear equations can be written in matrix form or as a 3x6 augmented matrix:

$$\begin{pmatrix} 4 & 3 & 1 & 0 & 0 \\ 2 & 1 & 0 & 1 & 0 \\ -70 & -50 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ P \end{pmatrix} = \begin{pmatrix} 240 \\ 100 \\ 0 \end{pmatrix} \quad \text{or} \quad \left( \begin{array}{ccccc|c} 4 & 3 & 1 & 0 & 0 & 240 \\ 2 & 1 & 0 & 1 & 0 & 100 \\ -70 & -50 & 0 & 0 & 1 & 0 \end{array} \right)$$

In the simplex method, the augmented matrix is referred to as the tableau.

The initial tableau is:

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
$s_1$	4	3	1	0	0	240
$s_2$	2	1	0	1	0	100
	-70	-50	0	0	1	0

The tableau represents the initial solution or vertex  $\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ P \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 240 \\ 100 \\ 0 \end{pmatrix}$ .

The slack variables  $s_1$  and  $s_2$  form the initial solution mix. The initial solution assumes that all available hours are unused i.e. the slack variables take the largest possible values.

Variables in the solution mix are called basic variables. Each basic variable has a column consisting of all 0's except for a single 1. All variables not in the solution mix take the value 0.

The simplex method uses a four step process (based on the Gauss Jordan method for solving a system of linear equations) to go from one tableau or vertex to the next. In this process, a basic variable in the solution mix is replaced by another variable previously not in the solution mix. The value of the replaced variable is set to 0.

Step 1

Select the pivot column (determine which variable to enter into the solution mix). Choose the column with the "most negative" element in the objective function row.

Basic variables	<i>x</i>	<i>y</i>	<i>s</i> <sub>1</sub>	<i>s</i> <sub>2</sub>	<i>P</i>	Righthand side
<i>s</i> <sub>1</sub>	4	3	1	0	0	240
<i>s</i> <sub>2</sub>	2	1	0	1	0	100
	<b>-70</b>	-50	0	0	1	0

*x* should enter into the solution mix because each unit of *x* (a table) contributes a profit of \$70 compared with only \$50 for each unit of *y* (a chair).

Step 2

Select the pivot row (determine which variable to replace in the solution mix). Divide the last element in each row by the corresponding element in the pivot column. The pivot row is the row with the smallest non-negative result.

Basic variables	<i>x</i>	<i>y</i>	<i>s</i> <sub>1</sub>	<i>s</i> <sub>2</sub>	<i>P</i>	Righthand side	
<i>s</i> <sub>1</sub>	4	3	1	0	0	240	240÷4 = 60
<i>s</i> <sub>2</sub>	2	1	0	1	0	100	<b>100÷2=50</b>
	<b>-70</b>	-50	0	0	1	0	

*s*<sub>2</sub> should be replaced by *x* in the solution mix. 60 tables can be made with 240 unused carpentry hours but only 50 tables can be made with the 100 unused finishing hours. Therefore we decide to make 50 tables.

Step 3

Calculate new values for the pivot row. Divide every number in the row by the pivot number.

R<sub>2</sub> / 2 :

	<i>x</i>	<i>y</i>	<i>s</i> <sub>1</sub>	<i>s</i> <sub>2</sub>	<i>P</i>	Righthand side
	4	3	1	0	0	240
	1	1/2	0	1/2	0	50
	-70	-50	0	0	1	0

### Step 4

Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number which remains as 1.

$R_1 - 4 \times R_2$  and  $R_3 + 70 \times R_2$  :

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
$s_1$	0	1	1	-2	0	40
$x$	1	1/2	0	1/2	0	50
	0	-15	0	35	1	3500

If 50 tables are made, then the unused carpentry hours are reduced by 200 hours (4 h/table multiplied by 50 tables); the value changes from 240 hours to 40 hours. Making 50 tables results in the profit being increased by \$3500 (\$70 per table multiplied by 50 tables); the value changes from \$0 to \$3500.

The new tableau represents the solution or vertex  $\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ P \end{pmatrix} = \begin{pmatrix} 50 \\ 0 \\ 40 \\ 0 \\ 3500 \end{pmatrix}$ .

The existence of 40 unused carpentry hours suggests that a more profitable solution can be found. For each table removed from the solution, 4 carpentry hours and 3 finishing hours are made available. If 2 unused carpentry hours are also taken from the 40 available, then 2 chairs can be made with the 6 carpentry hours and 3 finishing hours. Therefore, if 1 table is replaced by 2 chairs, the marginal increase in profit is \$30 (2 x \$50 less \$70).

Now repeat the steps until there are no negative numbers in the last row.

### Step 1

Select the pivot column.  $y$  should enter into the solution mix.

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
$s_1$	0	1	1	-2	0	40
$x$	1	1/2	0	1/2	0	50
	0	<b>-15</b>	0	35	1	3500

Each unit of  $y$  (a chair) added to the solution contributes a marginal increase in profit of \$15.

## Step 2

Select the pivot row.  $s_1$  should be replaced by  $y$  in the solution mix.

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side	
$s_1$	0	1	1	-2	0	40	<b><math>40 \div 1 = 40</math></b>
$x$	1	1/2	0	1/2	0	50	$50 \div 1/2 = 100$
	0	<b>-15</b>	0	35	1	3500	

40 chairs is the maximum number that can be made with the 40 unused carpentry hours.

## Step 3

Calculate new values for the pivot row. As the pivot number is already 1, there is no need to calculate new values for the pivot row.

## Step 4

Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number.

$R_2 - 1/2 \times R_1$  and  $R_3 + 15 \times R_1$  :

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
$y$	0	1	1	-2	0	40
$x$	1	0	-1/2	3/2	0	30
	0	0	15	5	1	4100

If 40 chairs are made, then the number of tables is reduced by 20 tables ( $1/2$  table/chair multiplied by 40 chairs); the value changes from 50 tables to 30 tables. The replacement of 20 tables by 40 chairs results in the profit being increased by \$600 (\$15 per chair multiplied by 40 chairs); the value changes from \$3100 to \$4100.

The new tableau represents the solution or vertex 
$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ P \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \\ 0 \\ 0 \\ 4100 \end{pmatrix}.$$

As the last row contains no negative numbers, this solution gives the maximum value of  $P$ . The maximum profit of \$4100 occurs when 30 tables and 40 chairs are made. There are no unused hours.

## APPENDIX A

### VOCABULARY

Linear programming - a mathematical technique that has the objective of maximising or minimising a quantity by choosing appropriate values for the variables involved. The objective and the constraints involved are expressed in terms of linear equations or inequalities. All the variables are restricted to taking non-negative values.

Objective function - a function that expresses the quantity to be maximised or minimised in terms of the other variables.

Constraint - a restriction that applies to the choice of values for the variables.

Standard maximization problem - a linear programming problem for which the objective function is to be maximized and all the constraints are "less-than-or-equal-to" inequalities.

Slack variable - a variable used to convert a "less-than-or-equal-to" inequality into an equation.

Augmented matrix or tableau - a matrix representing a system of linear equations.

Solution mix - the set of variables with non-zero values that form the solution to the problem.

Basic variable - a variable in the solution mix. The simplex method ensures that each basic variable has a column in the tableau consisting of all 0's except for a single 1.

Pivot column - the column of the tableau representing the variable to be entered into the solution mix.

Pivot row - the row of the tableau representing the variable to be replaced in the solution mix.

Pivot number - the element in both the pivot column and the pivot row.

### FOUR STEPS PROCESS FOR STANDARD MAXIMIZATION PROBLEM

1. Select the pivot column (the column with the "most negative" element in the objective function row).
2. Select the pivot row (the row with the smallest non-negative result when the last element in the row is divided by the corresponding element in the pivot column).
3. Calculate new values for the pivot row (divide every number in the row by the pivot number).
4. Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number.

APPENDIX B

RECOMMENDED COMMUNICATION AND JUSTIFICATION

The information for the Cannon Hill Furniture problem can be summarised:

	Tables	Chairs	Constraints	
Number produced per week	$x$	$y$	Cannot be negative	$x \geq 0, y \geq 0$
carpentry	4 h/table	3 h/chair	Maximum of 240 hours for the week	$4x + 3y \leq 240$
finishing	2 h/table	1 h/chair	Maximum of 100 hours for the week	$2x + y \leq 100$
Profit	\$70 per table	\$50 per chair		

The total profit (\$) for the week is given by the objective function  $P = 70x + 50y$ .

Introduce the slack variables:

$s_1$  unused carpentry hours in a week

$s_2$  unused finishing hours in a week.

The constraints become:

$$4x + 3y + 1s_1 + 0s_2 = 240$$

$$2x + 1y + 0s_1 + 1s_2 = 100$$

The slack variables can be included in the objective function:

$$P = 70x + 50y + 0s_1 + 0s_2$$

$$-70x - 50y + 0s_1 + 0s_2 + 1P = 0$$

The initial tableau is:

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side	
$s_1$	4	3	1	0	0	240	(240÷4 = 60)
$s_2$	2	1	0	1	0	100	(100÷2=50)
	<b>-70</b>	-50	0	0	1	0	

The pivot column is column 1 (-70 is the "most negative" element).

The pivot row is row 2 (50 is the smallest non-negative result).

$x$  replaces  $s_2$  in the solution mix.

$R_2 / 2$  :

	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
	4	3	1	0	0	240
	1	1/2	0	1/2	0	50
	<b>-70</b>	-50	0	0	1	0

$R_1 - 4 \times R_2$  and  $R_3 + 70 \times R_2$  :

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side	
$s_1$	0	1	1	-2	0	40	<b>(40+1=40)</b>
$x$	1	1/2	0	1/2	0	50	<b>(50+1/2=100)</b>
	0	<b>-15</b>	0	35	1	3500	

The pivot column is column 2 (-15 is the "most negative" element).

The pivot row is row 1 (40 is the smallest non-negative result).

$y$  replaces  $s_1$  in the solution mix.

As the pivot number is already 1, there is no need to calculate new values for the pivot row.

$R_2 - 1/2 \times R_1$  and  $R_3 + 15 \times R_1$  :

Basic variables	$x$	$y$	$s_1$	$s_2$	$P$	Righthand side
$y$	0	1	1	-2	0	40
$x$	1	0	-1/2	3/2	0	30
	0	0	15	5	1	4100

As the last row contains no negative numbers, this solution gives the maximum value of  $P$ .

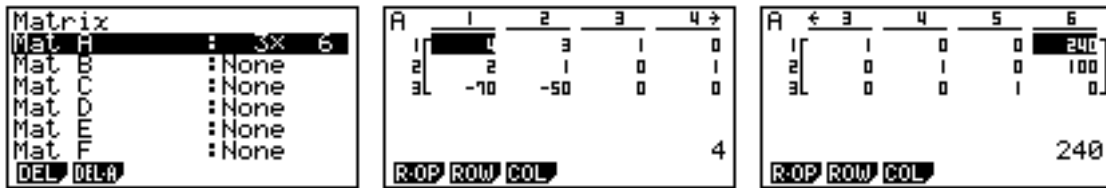
The tableau represents the solution

$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ P \end{pmatrix} = \begin{pmatrix} 30 \\ 40 \\ 0 \\ 0 \\ 4100 \end{pmatrix}.$$

The maximum profit of \$4100 occurs when 30 tables and 40 chairs are made. There are no unused hours.

APPENDIX C  
CASIO CALCULATOR - ROW OPERATIONS - MATRIX MODE

Consider the Cannon Hill Furniture problem. Enter the augmented matrix as matrix A.



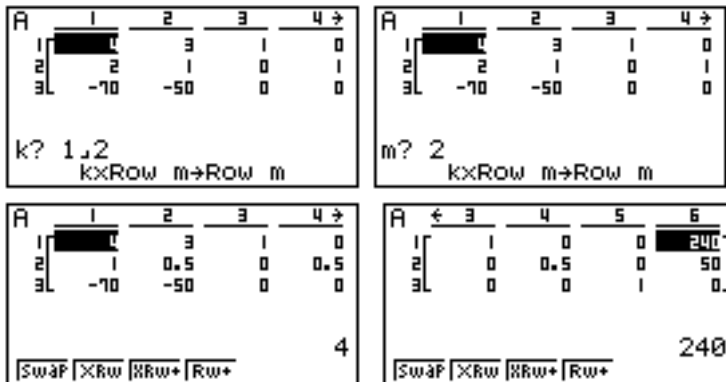
To save matrix A as matrix B in case you need to start again, in RUN mode:

- Press OPTN
- Select MAT (F2)
- Use Mat (F1)

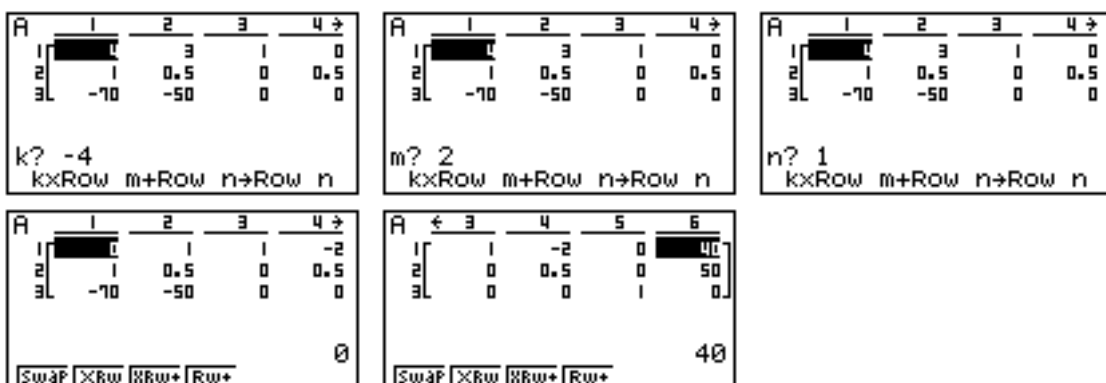


The arithmetic for row operations can be performed by using the calculator in MATRIX mode. Row operations ( $xRw$  and  $xRw+$ ) are available by pressing F1.

$R_2 / 2$  :



$R_1 - 4 \times R_2$  :



$R_3 + 70 \times R_2 :$

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$R_2 - 1/2 \times R_1 :$

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$R_3 + 15 \times R_1 :$

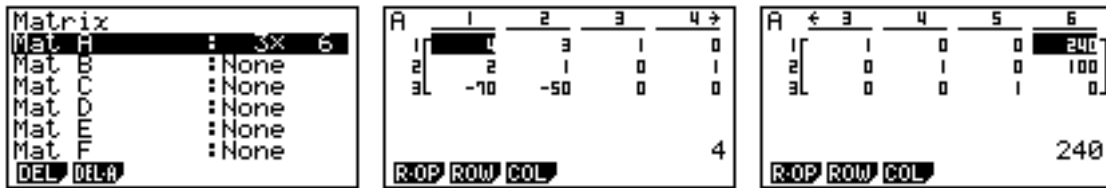
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Solution:  $P = 4100$ ,  $x = 30$  and  $y = 40$   
 30 tables and 40 chairs produce a maximum profit of \$4100.

Slack variables:  $s_1 = 0$  and  $s_2 = 0$   
 There are no unused hours.

APPENDIX D  
CASIO CALCULATOR - ROW OPERATIONS - PIVOT PROGRAM

Consider the Cannon Hill Furniture problem. Enter the augmented matrix as matrix A.



To save matrix A as matrix B in case you need to start again, in RUN mode:

- Press OPTN
- Select MAT (F2)
- Use Mat (F1)

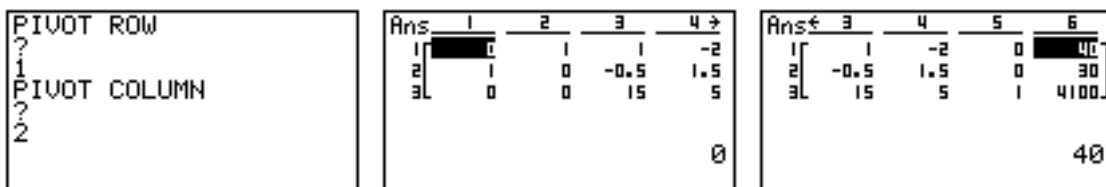


The arithmetic for row operations can be performed in the PIVOT program by entering the pivot row and pivot column.

$R_2 / 2$  and pivot on the element in row 2, column 1:



Pivot on the element in row 1, column 2:

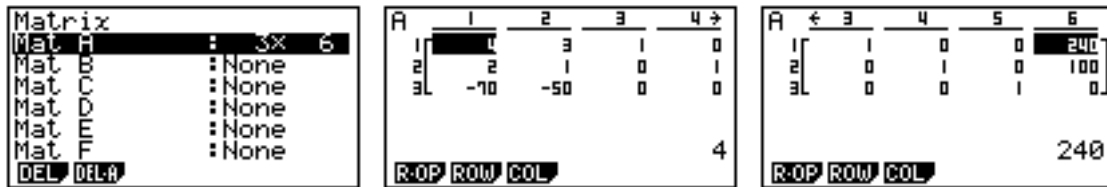


Solution:  $P = 4100$ ,  $x = 30$  and  $y = 40$   
30 tables and 40 chairs produce a maximum profit of \$4100.

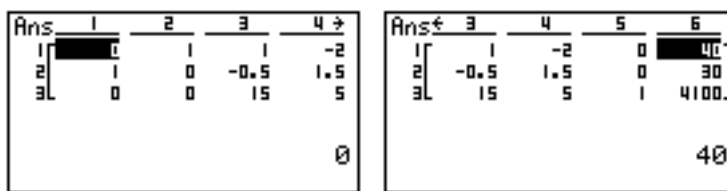
Slack variables:  $s_1 = 0$  and  $s_2 = 0$   
There are no unused hours.

APPENDIX E  
CASIO CALCULATOR - LINPROG PROGRAM (ALAN ZIV)  
STANDARD MAXIMIZATION PROBLEM

Consider the Cannon Hill Furniture problem. In MATRIX mode, enter the augmented matrix as matrix A.



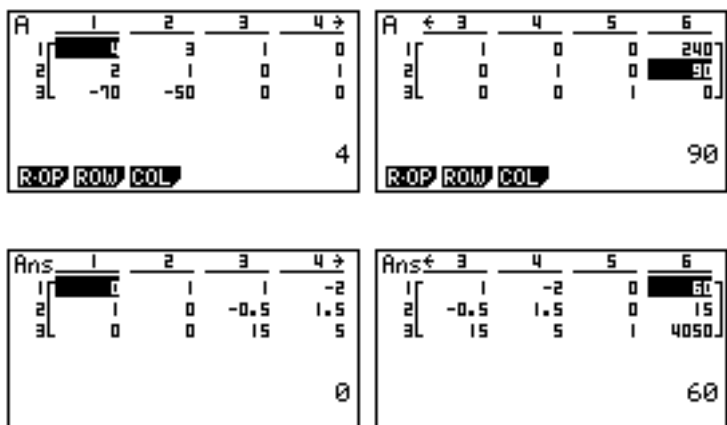
Run the program.



Solution:  $P = 4100$ ,  $x = 30$  and  $y = 40$   
30 tables and 40 chairs produce a maximum profit of \$4100.

Slack variables:  $s_1 = 0$  and  $s_2 = 0$   
There are no unused hours.

The initial tableau remains as matrix A. Changes can be made to the tableau and the program run again. For example, suppose the number of unused finishing hours is reduced from 100 to 90:



Solution:  $P = 4050$ ,  $x = 15$  and  $y = 60$   
15 tables and 60 chairs produce a maximum profit of \$4100.

Slack variables:  $s_1 = 0$  and  $s_2 = 0$   
There are no unused hours.

APPENDIX F - ONLINE SIMPLEX METHOD TOOL

[http://people.hofstra.edu/faculty/Stefan\\_Waner/RealWorld/simplex.html](http://people.hofstra.edu/faculty/Stefan_Waner/RealWorld/simplex.html)

<p>Enter the furniture problem in this format:</p> <p>Maximize <math>p = 70x + 50y</math> subject to  <math>4x + 3y \leq 240</math>  <math>2x + y \leq 100</math></p> <p>Solution:  <math>p = 4100</math>, <math>x = 30</math> and <math>y = 40</math></p> <p>30 tables and 40 chairs produce a maximum profit of \$4100.</p> <p>Slack variables:  <math>s_1 = 0</math> and <math>s_2 = 0</math></p> <p>There are no unused hours.</p>	<p>Tableau #1</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>4</td> <td>3</td> <td>1</td> <td>0</td> <td>0</td> <td>240</td> </tr> <tr> <td>2</td> <td>1</td> <td>0</td> <td>1</td> <td>0</td> <td>100</td> </tr> <tr> <td>-70</td> <td>-50</td> <td>0</td> <td>0</td> <td>1</td> <td>0</td> </tr> </tbody> </table> <p>Tableau #2</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>1</td> <td>-2</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>1/2</td> <td>0</td> <td>1/2</td> <td>0</td> <td>50</td> </tr> <tr> <td>0</td> <td>-15</td> <td>0</td> <td>35</td> <td>1</td> <td>3500</td> </tr> </tbody> </table> <p>Tableau #3</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>1</td> <td>-2</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>0</td> <td>-1/2</td> <td>3/2</td> <td>0</td> <td>30</td> </tr> <tr> <td>0</td> <td>0</td> <td>15</td> <td>5</td> <td>1</td> <td>4100</td> </tr> </tbody> </table>	x	y	s1	s2	p		4	3	1	0	0	240	2	1	0	1	0	100	-70	-50	0	0	1	0	x	y	s1	s2	p		0	1	1	-2	0	40	1	1/2	0	1/2	0	50	0	-15	0	35	1	3500	x	y	s1	s2	p		0	1	1	-2	0	40	1	0	-1/2	3/2	0	30	0	0	15	5	1	4100
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This tool is excellent for investigating scenarios. For example, what is the effect of introducing stools, each of which produces a profit of \$35 and requires one hour of carpentry and 1 hour of finishing?

<p>Maximize <math>p = 70x + 50y + 35z</math>  subject to  <math>4x + 3y + z \leq 240</math>  <math>2x + y + z \leq 100</math></p> <p>Solution:  <math>p = 4550</math>, <math>x = 0</math>, <math>y = 70</math>, <math>z = 30</math></p> <p>70 chairs and 30 stools produce a maximum profit of \$4550. No tables should be made.</p> <p>Slack variables:  <math>s_1 = 0</math> and <math>s_2 = 0</math></p> <p>There are no unused hours.</p>	<p>Tableau #1</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>4</td> <td>3</td> <td>1</td> <td>1</td> <td>0</td> <td>0</td> <td>240</td> </tr> <tr> <td>2</td> <td>1</td> <td>1</td> <td>0</td> <td>1</td> <td>0</td> <td>100</td> </tr> <tr> <td>-70</td> <td>-50</td> <td>-35</td> <td>0</td> <td>0</td> <td>1</td> <td>0</td> </tr> </tbody> </table> <p>Tableau #2</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>-1</td> <td>1</td> <td>-2</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>1/2</td> <td>1/2</td> <td>0</td> <td>1/2</td> <td>0</td> <td>50</td> </tr> <tr> <td>0</td> <td>-15</td> <td>0</td> <td>0</td> <td>35</td> <td>1</td> <td>3500</td> </tr> </tbody> </table> <p>Tableau #3</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>-1</td> <td>1</td> <td>-2</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>0</td> <td>1</td> <td>-1/2</td> <td>3/2</td> <td>0</td> <td>30</td> </tr> <tr> <td>0</td> <td>0</td> <td>-15</td> <td>15</td> <td>5</td> <td>1</td> <td>4100</td> </tr> </tbody> </table> <p>Tableau #4</p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>1</td> <td>1</td> <td>0</td> <td>1/2</td> <td>-1/2</td> <td>0</td> <td>70</td> </tr> <tr> <td>1</td> <td>0</td> <td>1</td> <td>-1/2</td> <td>3/2</td> <td>0</td> <td>30</td> </tr> <tr> <td>15</td> <td>0</td> <td>0</td> <td>15/2</td> <td>55/2</td> <td>1</td> <td>4550</td> </tr> </tbody> </table>	x	y	z	s1	s2	p		4	3	1	1	0	0	240	2	1	1	0	1	0	100	-70	-50	-35	0	0	1	0	x	y	z	s1	s2	p		0	1	-1	1	-2	0	40	1	1/2	1/2	0	1/2	0	50	0	-15	0	0	35	1	3500	x	y	z	s1	s2	p		0	1	-1	1	-2	0	40	1	0	1	-1/2	3/2	0	30	0	0	-15	15	5	1	4100	x	y	z	s1	s2	p		1	1	0	1/2	-1/2	0	70	1	0	1	-1/2	3/2	0	30	15	0	0	15/2	55/2	1	4550
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If stools are introduced, what is the effect of insisting that at least four chairs are produced for each table?

<p><b>Maximize</b>  <math>p = 70x + 50y + 35z</math>  <b>subject to</b>  <math>4x + 3y + z \leq 240</math>  <math>2x + y + z \leq 100</math>  <math>y - 4x \leq 0</math></p> <p><b>Solution:</b>  <math>p = 4340, x = 14, y = 56,</math>  <math>z = 16</math></p> <p>14 tables, 56 chairs and  14 stools produce a  maximum profit of \$4340.</p> <p><b>Slack variables:</b>  <math>s1 = 0, s2 = 0</math> and <math>s3 = 0</math></p> <p>There are no unused hours.</p>	<p><b>Tableau #1</b></p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>s3</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>4</td> <td>3</td> <td>1</td> <td>1</td> <td>0</td> <td>0</td> <td>0</td> <td>240</td> </tr> <tr> <td>2</td> <td>1</td> <td>1</td> <td>0</td> <td>1</td> <td>0</td> <td>0</td> <td>100</td> </tr> <tr> <td>-4</td> <td>1</td> <td>0</td> <td>0</td> <td>0</td> <td>1</td> <td>0</td> <td>0</td> </tr> <tr> <td>-70</td> <td>-50</td> <td>-35</td> <td>0</td> <td>0</td> <td>0</td> <td>1</td> <td>0</td> </tr> </tbody> </table> <p><b>Tableau #2</b></p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>s3</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>-1</td> <td>1</td> <td>-2</td> <td>0</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>1/2</td> <td>1/2</td> <td>0</td> <td>1/2</td> <td>0</td> <td>0</td> <td>50</td> </tr> <tr> <td>0</td> <td>3</td> <td>2</td> <td>0</td> <td>2</td> <td>1</td> <td>0</td> <td>200</td> </tr> <tr> <td>0</td> <td>-15</td> <td>0</td> <td>0</td> <td>35</td> <td>0</td> <td>1</td> <td>3500</td> </tr> </tbody> </table> <p><b>Tableau #3</b></p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>s3</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>-1</td> <td>1</td> <td>-2</td> <td>0</td> <td>0</td> <td>40</td> </tr> <tr> <td>1</td> <td>0</td> <td>1</td> <td>-1/2</td> <td>3/2</td> <td>0</td> <td>0</td> <td>30</td> </tr> <tr> <td>0</td> <td>0</td> <td>5</td> <td>-3</td> <td>8</td> <td>1</td> <td>0</td> <td>80</td> </tr> <tr> <td>0</td> <td>0</td> <td>-15</td> <td>15</td> <td>5</td> <td>0</td> <td>1</td> <td>4100</td> </tr> </tbody> </table> <p><b>Tableau #4</b></p> <table border="1"> <thead> <tr> <th>x</th> <th>y</th> <th>z</th> <th>s1</th> <th>s2</th> <th>s3</th> <th>p</th> <th></th> </tr> </thead> <tbody> <tr> <td>0</td> <td>1</td> <td>0</td> <td>2/5</td> <td>-2/5</td> <td>1/5</td> <td>0</td> <td>56</td> </tr> <tr> <td>1</td> <td>0</td> <td>0</td> <td>1/10</td> <td>-1/10</td> <td>-1/5</td> <td>0</td> <td>14</td> </tr> <tr> <td>0</td> <td>0</td> <td>1</td> <td>-3/5</td> <td>8/5</td> <td>1/5</td> <td>0</td> <td>16</td> </tr> <tr> <td>0</td> <td>0</td> <td>0</td> <td>6</td> <td>29</td> <td>3</td> <td>1</td> <td>4340</td> </tr> </tbody> </table>	x	y	z	s1	s2	s3	p		4	3	1	1	0	0	0	240	2	1	1	0	1	0	0	100	-4	1	0	0	0	1	0	0	-70	-50	-35	0	0	0	1	0	x	y	z	s1	s2	s3	p		0	1	-1	1	-2	0	0	40	1	1/2	1/2	0	1/2	0	0	50	0	3	2	0	2	1	0	200	0	-15	0	0	35	0	1	3500	x	y	z	s1	s2	s3	p		0	1	-1	1	-2	0	0	40	1	0	1	-1/2	3/2	0	0	30	0	0	5	-3	8	1	0	80	0	0	-15	15	5	0	1	4100	x	y	z	s1	s2	s3	p		0	1	0	2/5	-2/5	1/5	0	56	1	0	0	1/10	-1/10	-1/5	0	14	0	0	1	-3/5	8/5	1/5	0	16	0	0	0	6	29	3	1	4340
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## APPENDIX G - CONSTRAINTS OTHER THAN "LESS-THAN-OR-EQUAL-TO" INEQUALITIES

Consider the problem of finding the maximum value of  $p = 2x + y$ , subject to the constraints:

$$x + y \geq 10$$

$$3x + y \geq 15$$

$$y \leq 12$$

$$x \leq 8$$

$$x \geq 0, y \geq 0$$

This is not a standard maximisation problem because two of the constraints are not "less-than-or-equal-to" inequalities.

Consider the first inequality:

$$x + y \geq 10$$

Convert to an equation by introducing a surplus variable  $s_1$ :

$$x + y - s_1 = 10$$

Using the same approach for the second inequality, the constraints become:

$$1x + 1y - 1s_1 + 0s_2 + 0s_3 + 0s_4 = 10$$

$$3x + 1y + 0s_1 - 1s_2 + 0s_3 + 0s_4 = 15$$

$$0x + 1y + 0s_1 + 0s_2 + 1s_3 + 0s_4 = 12$$

$$1x + 0y + 0s_1 + 0s_2 + 0s_3 + 1s_4 = 8$$

The objective function becomes:

$$p = 2x + 1y + 0s_1 + 0s_2 + 0s_3 + 0s_4$$

$$-2x - 1y + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 1p = 0$$

The initial tableau is:

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>s1</b>	1	1	-1	0	0	0	0	10
<b>s2</b>	3	1	0	-1	0	0	0	15
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	1	0	0	0	0	1	0	8
	-2	-1	0	0	0	0	1	0

This tableau represents the solution

$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -10 \\ -15 \\ 12 \\ 8 \\ 0 \end{pmatrix}.$$

Linear programming requires that all the variables be non-negative. To make this happen, consider only those rows for variables with negative values (1<sup>st</sup> and 2<sup>nd</sup> rows) and perform the following:

- Select the pivot row (the row with the largest value in the last column).
- Select the pivot column (the column with the largest value in the pivot row, ignoring the last column).
- Calculate new values for the pivot row (divide every number in the row by the pivot number).
- Use row operations to make all numbers in the pivot column equal to 0 except for the pivot number.

Repeat the above until all the variables are non-negative.

Pivot row is row 2 (15 > 10).

Pivot column is column 1 (3 is the largest value in the pivot row, ignoring the last column).

$R_2 / 3$  :

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>s1</b>	1	1	-1	0	0	0	0	10
<b>s2</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	1	0	0	0	0	1	0	8
	-2	-1	0	0	0	0	1	0

$R_1 - R_2, R_3 - R_2, R_4 + 2 \times R_2$  :

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>s1</b>	0	2/3	-1	1/3	0	0	0	5
<b>x</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	0	-1/3	0	1/3	0	1	0	3
	0	-1/3	0	-2/3	0	0	1	10

This tableau represents the solution 
$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ p \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \\ -5 \\ 0 \\ 12 \\ 3 \\ 10 \end{pmatrix} .$$

Pivot row is row 1 (only row with a negative value for the variable).

Pivot column is column 2 (2/3 is the largest value in the pivot row, ignoring the last column).

$R_1 \times 3/2 :$

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>s1</b>	0	1	-3/2	1/2	0	0	0	15/2
<b>x</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	0	-1/3	0	1/3	0	1	0	3
	0	-1/3	0	-2/3	0	0	1	10

$R_2 - 1/3 \times R_1, R_3 - R_1, R_4 + 1/3 \times R_1, R_5 + 1/3 \times R_1 :$

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>y</b>	0	1	-3/2	1/2	0	0	0	15/2
<b>x</b>	1	0	1/2	-1/2	0	0	0	5/2
<b>s3</b>	0	0	3/2	-1/2	1	0	0	9/2
<b>s4</b>	0	0	-1/2	1/2	0	1	0	11/2
	0	0	-1/2	-1/2	0	0	1	25/2

This tableau represents the solution

$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ p \end{pmatrix} = \begin{pmatrix} 5/2 \\ 15/2 \\ 0 \\ 0 \\ 9/2 \\ 11/2 \\ 25/2 \end{pmatrix}.$$

As all the variables are non-negative, this solution is a vertex in a standard maximisation problem and we can now use repeated application of the four steps process.

The pivot column is column 3 (-1/2 is the "most negative" element, choose column 3 or 4). The pivot row is row 3 ( $5/2 \div 1/2 = 5$ ,  $9/2 \div 3/2 = 3$ , 3 is the smallest non-negative result).  $s_1$  replaces  $s_3$  in the solution mix.

$R_3 \times 2/3 :$

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>y</b>	0	1	-3/2	1/2	0	0	0	15/2
<b>x</b>	1	0	1/2	-1/2	0	0	0	5/2
<b>s3</b>	0	0	1	-1/3	2/3	0	0	3
<b>s4</b>	0	0	-1/2	1/2	0	1	0	11/2
	0	0	-1/2	-1/2	0	0	1	25/2

$R_1 + 3/2 \times R_3, R_2 - 1/2 \times R_3, R_4 + 1/2 \times R_3, R_5 + 1/2 \times R_3 :$

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>y</b>	0	1	0	0	1	0	0	12
<b>x</b>	1	0	0	-1/3	-1/3	0	0	1
<b>s1</b>	0	0	1	-1/3	2/3	0	0	3
<b>s4</b>	0	0	0	1/3	1/3	1	0	7
	0	0	0	-2/3	1/3	0	1	14

The pivot column is column 4 (-2/3 is the "most negative" element).  
 The pivot row is row 4 ( $7 \div 1/3 = 21$  is the only non-negative result).  
 $s_2$  replaces  $s_4$  in the solution mix.

$R_4 \times 3$  :

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>y</b>	0	1	0	0	1	0	0	12
<b>x</b>	1	0	0	-1/3	-1/3	0	0	1
<b>s1</b>	0	0	1	-1/3	2/3	0	0	3
<b>s4</b>	0	0	0	1	1	3	0	21
	0	0	0	-2/3	1/3	0	1	14

$R_2 + 1/3 \times R_4$ ,  $R_3 + 1/3 \times R_4$ ,  $R_5 + 2/3 \times R_4$  :

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>p</b>	<b>rhs</b>
<b>y</b>	0	1	0	0	1	0	0	12
<b>x</b>	1	0	0	0	0	1	0	8
<b>s1</b>	0	0	1	0	1	1	0	10
<b>s2</b>	0	0	0	1	1	3	0	21
	0	0	0	0	1	2	1	28

This tableau represents the solution

$$\begin{pmatrix} x \\ y \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ p \end{pmatrix} = \begin{pmatrix} 8 \\ 12 \\ 10 \\ 21 \\ 0 \\ 0 \\ 28 \end{pmatrix} .$$

As the last row of the tableau contains no negative numbers, this solution gives the maximum value  $p = 28$ , occurring when  $x = 8$  and  $y = 12$ .

Note:

The approach used by the simplex method tool in Appendix F appears similar to the above. Other approaches are possible!

## APPENDIX H - MINIMISATION PROBLEM

Consider the problem as that in Appendix G but find the minimum value of  $p = 2x + y$ , subject to the constraints:

$$\begin{aligned} x + y &\geq 10 \\ 3x + y &\geq 15 \\ y &\leq 12 \\ x &\leq 8 \\ x \geq 0, y &\geq 0 \end{aligned}$$

This is not a standard maximisation problem because:

- it is a minimisation problem
- two of the constraints are not "less-than-or-equal-to" inequalities.

As in Appendix G, the constraints become:

$$\begin{aligned} 1x + 1y - 1s_1 + 0s_2 + 0s_3 + 0s_4 &= 10 \\ 3x + 1y + 0s_1 - 1s_2 + 0s_3 + 0s_4 &= 15 \\ 0x + 1y + 0s_1 + 0s_2 + 1s_3 + 0s_4 &= 12 \\ 1x + 0y + 0s_1 + 0s_2 + 0s_3 + 1s_4 &= 8 \end{aligned}$$

Change the problem to a maximisation problem by finding the maximum value of:

$$c = -p = -2x - y$$

The objective function becomes:

$$\begin{aligned} c &= -2x - 1y + 0s_1 + 0s_2 + 0s_3 + 0s_4 \\ 2x + 1y + 0s_1 + 0s_2 + 0s_3 + 0s_4 + 1c &= 0 \end{aligned}$$

The initial tableau is:

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>c=-p</b>	<b>rhs</b>
<b>s1</b>	1	1	-1	0	0	0	0	10
<b>s2</b>	3	1	0	-1	0	0	0	15
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	1	0	0	0	0	1	0	8
	2	1	0	0	0	0	1	0

Using the approach of Appendix G gives the following tableaux. Add your own justification!

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>c=-p</b>	<b>rhs</b>
<b>s1</b>	1	1	-1	0	0	0	0	10
<b>s2</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	1	0	0	0	0	1	0	8
	2	1	0	0	0	0	1	0

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>c=-p</b>	<b>rhs</b>
<b>s1</b>	0	2/3	-1	1/3	0	0	0	5
<b>x</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	0	-1/3	0	1/3	0	1	0	3
	0	1/3	0	2/3	0	0	1	-10

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>c=-p</b>	<b>rhs</b>
<b>s1</b>	0	1	-3/2	1/2	0	0	0	15/2
<b>x</b>	1	1/3	0	-1/3	0	0	0	5
<b>s3</b>	0	1	0	0	1	0	0	12
<b>s4</b>	0	-1/3	0	1/3	0	1	0	3
	0	1/3	0	2/3	0	0	1	-10

	<b>x</b>	<b>y</b>	<b>s1</b>	<b>s2</b>	<b>s3</b>	<b>s4</b>	<b>c=-p</b>	<b>rhs</b>
<b>y</b>	0	1	-3/2	1/2	0	0	0	15/2
<b>x</b>	1	0	1/2	-1/2	0	0	0	5/2
<b>s3</b>	0	0	3/2	-1/2	1	0	0	9/2
<b>s4</b>	0	0	-1/2	1/2	0	1	0	11/2
	0	0	1/2	1/2	0	0	1	-25/2

The minimum value is  $p = 25/2$  and this occurs when  $x = 5/2$  and  $y = 15/2$ .

Note:

The approach used by the simplex method tool in Appendix F appears similar to the above. Other approaches are possible!