MATRICES - INTRODUCTION

- A *matrix* is a rectangular array of numbers called *elements.* The array is enclosed by brackets.

- Matrices are usually denoted by capital letters eg. $\mathbf{A}=\begin{pmatrix} 4 & 9 \\ -2 & 4\cdot 75 \\ 0 & 1 \end{pmatrix}$

- The *order* or size of a matrix is written as $m \times n$ where m is the number of *rows* and n is the number of *columns*.

  eg. $\begin{pmatrix} 4 & -2 & 1 \\ 0 & 5 & -8 \end{pmatrix}$ is a 2 x 3 matrix

- A *row matrix* or *row vector* is a matrix with only one row eg. $\begin{pmatrix} -4 & 6\cdot 3 & 8 \end{pmatrix}$.

- A *column matrix* or *column vector* is a matrix with only one column eg. $\begin{pmatrix} 3 \\ -6 \\ 2 \end{pmatrix}$.

- A *square matrix* has the same number of rows as columns.

  A $n \times n$ square matrix is said to be of *order* n.

  eg. $\begin{pmatrix} 5 & 9 \\ -2 & 3\tfrac{1}{2} \end{pmatrix}$ is a square matrix of order 2.

- The elements of a matrix can be identified by subscripts.

  eg. a_{21} is the element in the 2nd row and 1st column.

  eg. consider the 3 x 3 matrix $\mathbf{A}=(a_{ij})$ or $\mathbf{A}=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

- The *zero matrix* is a matrix with every element equal to zero.

  eg. $\mathbf{0}_{2\times 3}=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ is the zero matrix of order 2 x 3

  eg. $\mathbf{0}_{2\times 2}=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is the zero matrix of order 2

- The *identity matrix* is a square matrix with all elements on the *leading diagonal* equal to 1 and all other elements equal to 0.

  eg. $\mathbf{I}_3=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the identity matrix of order 3

- The transpose of a matrix is the matrix obtained by interchanging the rows and columns.

  eg. if $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{pmatrix}$ then the transpose $\mathbf{B}'$ or $\mathbf{B}^{\mathrm{T}} = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 2 & 4 & 6 & 8 \end{pmatrix}$

  NB. The element 6 is in row 3 and column 2 of $\mathbf{B}$.
  The element 6 is in row 2 and column 3 of $\mathbf{B}^{\mathrm{T}}$.

  For any matrix $\mathbf{A} = (a_{ij})$, we can write $\mathbf{A}^{\mathrm{T}} = (a_{ji})$

☺ If $\mathbf{A}^{\mathrm{T}} = \mathbf{A}$, what can you say about a matrix $\mathbf{A}$?

☺ If matrices $\mathbf{A}$ and $\mathbf{B}$ have the same order, show that $(\mathbf{A}+\mathbf{B})^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}} + \mathbf{B}^{\mathrm{T}}$.

- Matrices can be used to store information.

  Eg. The stocks of 4 types of walkmans held at the beginning of May at 3 different stores could be recorded in a 3 x 4 matrix $\mathbf{T}$.

  $\mathbf{T} = \begin{pmatrix} 12 & 7 & 9 & 4 \\ 3 & 6 & 6 & 5 \\ 8 & 2 & 5 & 1 \end{pmatrix}$ ie. in store 3, there are 8 walkmans of type 1.

  If we wished, we could label the rows S_1, S_2, S_3 to represent the stores and the columns W_1, W_2, W_3, W_4 to represent the walkmans.

- Two matrices are said to be *equal* if they are of the same order and if all the corresponding elements are equal.

- *Addition* (or *subtraction*) of matrices is possible if they are of the same order. The corresponding elements are added (or subtracted).

  In the walkman example, suppose the sales during May are given by $\mathbf{M}$ and the deliveries during the month are given by $\mathbf{D}$.

  $\mathbf{M} = \begin{pmatrix} 5 & 7 & 6 & 1 \\ 6 & 3 & 5 & 4 \\ 6 & 1 & 2 & 3 \end{pmatrix}$ and $\mathbf{D} = \begin{pmatrix} 2 & 2 & 1 & 3 \\ 3 & 2 & 0 & 0 \\ 4 & 2 & 1 & 5 \end{pmatrix}$

  $\mathbf{T} + \mathbf{D} - \mathbf{M}$

  $$= \begin{pmatrix} 12 & 7 & 9 & 4 \\ 3 & 6 & 6 & 5 \\ 8 & 2 & 5 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 2 & 1 & 3 \\ 3 & 2 & 0 & 0 \\ 4 & 2 & 1 & 5 \end{pmatrix} - \begin{pmatrix} 5 & 7 & 6 & 1 \\ 6 & 3 & 5 & 4 \\ 6 & 1 & 2 & 3 \end{pmatrix}$$

  $$= \begin{pmatrix} 9 & 2 & 4 & 6 \\ 0 & 5 & 1 & 1 \\ 6 & 3 & 4 & 3 \end{pmatrix}$$

☺ What is represented by the matrix $\mathbf{U} = \mathbf{T} + \mathbf{D} - \mathbf{M}$?

■ Matrices of the same order form a group under addition ie. the closure, associative, identity and inverse properties hold. For $m \times n$ matrices, the additive identity is the zero matrix $\mathbf{0}_{m \times n}$.

■ A matrix can be *multiplied by a scalar*. Each element is multiplied by the number.

$$12\mathbf{M} = 12\begin{pmatrix} 5 & 7 & 6 & 1 \\ 6 & 3 & 5 & 4 \\ 6 & 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 60 & 84 & 72 & 12 \\ 72 & 36 & 60 & 48 \\ 72 & 12 & 24 & 36 \end{pmatrix}$$

☺ What is represented by the matrix $12\mathbf{M}$?

☺ For any matrix $\mathbf{A}$ and scalar r $(r \neq 0)$, show that $(r\,\mathbf{A})^{\mathrm{T}} = r\,\mathbf{A}^{\mathrm{T}}$.

■ *Two matrices can be multiplied* if the number of columns in the first matrix equals the number of rows in the second matrix. Matrices that fulfil this condition are called *conformable*.

In the walkman example, suppose the selling prices are given by the matrix $\mathbf{P} = \begin{pmatrix} 90 \\ 70 \\ 50 \\ 30 \end{pmatrix}$.

The total values of sales during May for each store are given by the matrix product **MP**.

MP

$$= \begin{pmatrix} 5 & 7 & 6 & 1 \\ 6 & 3 & 5 & 4 \\ 6 & 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 90 \\ 70 \\ 50 \\ 30 \end{pmatrix}$$

$$= \begin{pmatrix} 5\times90+7\times70+6\times50+1\times30 \\ 6\times90+3\times70+5\times50+4\times30 \\ 6\times90+1\times70+2\times50+3\times30 \end{pmatrix}$$

$$= \begin{pmatrix} 450+490+300+30 \\ 540+210+250+120 \\ 540+70+100+90 \end{pmatrix}$$

$$= \begin{pmatrix} 1270 \\ 1120 \\ 800 \end{pmatrix}$$

Suppose the selling price and the profit for each walkman are recorded in a single matrix:

$$\mathbf{Q} = \begin{pmatrix} 90 & 30 \\ 70 & 25 \\ 50 & 20 \\ 30 & 10 \end{pmatrix}$$

The total values of sales and the total profits during May for each store are given by the matrix product $\mathbf{MQ}$.

$$\mathbf{MQ}$$

$$= \begin{pmatrix} 5 & 7 & 6 & 1 \\ 6 & 3 & 5 & 4 \\ 6 & 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 90 & 30 \\ 70 & 25 \\ 50 & 20 \\ 30 & 10 \end{pmatrix}$$

$$= \begin{pmatrix} 5\times90+7\times70+6\times50+1\times30 & 5\times30+7\times25+6\times20+1\times10 \\ 6\times90+3\times70+5\times50+4\times30 & 6\times30+3\times25+5\times20+4\times10 \\ 6\times90+1\times70+2\times50+3\times30 & 6\times30+1\times25+2\times20+3\times10 \end{pmatrix}$$

$$= \begin{pmatrix} 1270 & 455 \\ 1120 & 395 \\ 800 & 275 \end{pmatrix}$$

■ To determine whether matrix multiplication is possible and the order of the result, consider the orders of the matrices involved:

matrix	**M**	**Q**	**MQ**
order	3 x 4	4 x 2	3 x 2
	middle numbers the same (both 4)		take first and last number

☺ In the walkman example, why is not possible to calculate **QM**?

☺ If **CD** and **DC** can both be calculated, what can you say about **C** and **D**?

☺ Show that *matrix multiplication is not commutative* for square matrices ie. find a counter example, square matrices **E** and **F** such that $\mathbf{EF} \neq \mathbf{FE}$?

☺ If **G** has order $m \times n$ and **H** has order $n \times m$, is it possible to calculate **GH** and **HG** and what are the orders of the products?

■ Matrix multiplication is *associative* ie. $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$. Check this property for 3 matrices of your own choosing.

■ The *distributive* property applies to matrix addition and multiplication ie.

$$\mathbf{A}(\mathbf{B}+\mathbf{C}) = \mathbf{AB}+\mathbf{AC} \quad \text{and} \quad (\mathbf{B}+\mathbf{C})\mathbf{A} = \mathbf{BA}+\mathbf{CA}$$

Check this property for 3 matrices of your own choosing.

- ☺ Show that $\mathbf{I}_3$ is the identity under multiplication for 3×3 matrices.
- ☺ Show that for any 2×2 matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $(\mathbf{AB})^{\mathsf{T}}=\mathbf{B}^{\mathsf{T}}\mathbf{A}^{\mathsf{T}}$.
- ☺ If $\mathbf{A}$ is any 2×2 matrix, show that $\mathbf{A}\,\mathbf{A}^{\mathsf{T}}$ is a *symmetric* matrix ie. symmetric about the leading diagonal.