11 MATHEMATICS C - REVIEW SHEET

1) What is meant by:
   a) the order of a matrix
   b) a row matrix
   c) a column matrix
   d) a square matrix?

2) State whether each of the following is true or false.
   a) Two matrices can only be added if they are of the same order.
   b) Two matrices can only be multiplied if they are of the same order.
   c) Addition of matrices is always associative.
   d) Addition of matrices is always commutative.
   e) Multiplication of matrices is always associative.
   f) Multiplication of matrices is always commutative.
   g) If **CD** and **DC** can both be calculated, then **C** and **D** are both square.
   h) If **C** and **D** are both square, then **CD** and **DC** can both be calculated.
   i) The *distributive* property applies to matrix addition and multiplication ie.
   $$\mathbf{A}(\mathbf{B}+\mathbf{C})=\mathbf{AB}+\mathbf{AC} \quad \text{and} \quad (\mathbf{B}+\mathbf{C})\mathbf{A}=\mathbf{BA}+\mathbf{CA}$$

3) Consider the matrices below.

$$\mathbf{A}=\begin{pmatrix}1 & 3\\ 4 & 2\\ 0 & -1\\ 1 & 0\end{pmatrix} \quad \mathbf{B}=\begin{pmatrix}3 & 0\\ -2 & 1\\ 4 & 2\end{pmatrix} \quad \mathbf{C}=\begin{pmatrix}1 & 1 & 0 & 0 & 1\\ 2 & 0 & 2 & 0 & 0\\ 3 & -5 & 1 & 0 & 3\\ 4 & 7 & -2 & 0 & 1\end{pmatrix} \quad \mathbf{D}=\begin{pmatrix}5 & 7\\ -4 & 2\\ 5 & 3\\ -1 & 0\end{pmatrix} \quad \mathbf{E}=\begin{pmatrix}2 & 1 & -1\end{pmatrix} \quad \mathbf{F}=\begin{pmatrix}-3 & 2\end{pmatrix}$$

   a) What is the value of c_{32}?
   b) Write down $\mathbf{C}^{\mathrm{T}}$.
   c) Evaluate $3\mathbf{D}$.
   d) Choose two matrices from those provided and add them.
   e) Choose two matrices from those provided and multiply them.
   f) Evaluate $\mathbf{I}_3\mathbf{B}$.
   g) Evaluate $\begin{pmatrix}1 & 1 & 1\end{pmatrix}\mathbf{B}\begin{pmatrix}1\\ 1\end{pmatrix}$

4) If $\mathbf{D}=\begin{pmatrix}1 & -1 & 1 & -1\\ 1 & 1 & -1 & -1\end{pmatrix}$, evaluate $\mathbf{D}^{\mathrm{T}}\mathbf{D}$ and $\mathbf{D}\mathbf{D}^{\mathrm{T}}$.

5) If $\mathbf{A}=\begin{pmatrix}2 & 4\\ -3 & 1\end{pmatrix}$, $\mathbf{B}=\begin{pmatrix}-2 & 5\\ 6 & -1\end{pmatrix}$ and $\mathbf{C}=\begin{pmatrix}1 & -2\\ 7 & 0\end{pmatrix}$:
   a) evaluate $\mathbf{A}^2$
   b) evaluate $\mathbf{C}^3$
   c) solve the equation $4\mathbf{A}+3\mathbf{X}=\mathbf{B}+2\mathbf{C}^{\mathrm{T}}-2\mathbf{X}$

6) In what way can all 4 x 2 matrices form a group? What is the identity? Given a 4 x 2 matrix, how do you find its inverse?

7) If $\mathbf{P}=\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ and $\mathbf{Q}=\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$, show that **P** and **Q** are commutative matrices under multiplication.

8) Show that for any 2 x 2 matrices $\mathbf{A}=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\mathbf{B}=\begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $(\mathbf{AB})^{\mathrm{T}}=\mathbf{B}^{\mathrm{T}}\mathbf{A}^{\mathrm{T}}$.

9) Show that the following set of matrices forms a group under multiplication.

$$\left\{\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} : k \in R\,,\, k \neq 0\right\}$$

10) The long-term population growth rate for a population is estimated to be 40% pa. Calculate a suitable culling factor to achieve:
   a) a stable population level
   b) a long-term population growth rate of 15% pa.

11) The female population of a species is shown in the table below together with estimates of breeding and survival rates.

age (years)	0 - 2	2 - 4	4 - 6	6 - 8	8 - 10
initial population	1800	1500	1150	700	400
breeding rate	0	0·4	1·5	1·2	0·3
survival rate	0·6	0·9	0·7	0·5	0

   a) Write down the Leslie matrix **L**.
   b) Estimate the total population in 6 years.
   c) If 20% of the population is harvested every two years, estimate the total population in 10 years.
   d) If 30% of 2 - 4 year olds are harvested every two years, estimate the total population in 10 years.
   e) If no harvesting takes place, estimate the long-term population growth rate.

ANSWERS

(2) (a) T (b) F (c) T (d) T (e) T (f) F (g) F (h) F (i) T

(3) (a) $^{-}5$ (b) $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & ^{-}5 & 7 \\ 0 & 2 & 1 & ^{-}2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$ (c) $\begin{pmatrix} 15 & 21 \\ ^{-}12 & 6 \\ 15 & 9 \\ ^{-}3 & 0 \end{pmatrix}$ (f) **B** (g) (8)

(4) (a) $\begin{pmatrix} 2 & 0 & 0 & ^{-}2 \\ 0 & 2 & ^{-}2 & 0 \\ 0 & ^{-}2 & 2 & 0 \\ ^{-}2 & 0 & 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ (5) (a) $\begin{pmatrix} ^{-}8 & 12 \\ ^{-}9 & ^{-}11 \end{pmatrix}$ (b) $\begin{pmatrix} ^{-}27 & 26 \\ ^{-}91 & ^{-}14 \end{pmatrix}$ (c) $\begin{pmatrix} ^{-}\frac{8}{5} & \frac{3}{5} \\ \frac{14}{5} & ^{-}1 \end{pmatrix}$

(6) 4 x 2 matrices form a group under addition.

The identity is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$. To find the inverse of a matrix, change the sign of each element.

(10) (a) 28·6% pa (b) 17·9% pa

(11) (a) $\begin{pmatrix} 0 & 0\cdot4 & 1\cdot5 & 1\cdot2 & 0\cdot3 \\ 0\cdot6 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot9 & 0 & 0 & 0 \\ 0 & 0 & 0\cdot7 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot5 & 0 \end{pmatrix}$ (b) 17090 (c) 7743 (d) 13741 (e) 15·1% every two years

(3)

(a) $c_{32} = {}^{-}5$ (element in 3rd row and 2nd column)

(b)
$$\underset{\sim}{C}^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 0 & -5 & 7 \\ 0 & 2 & 1 & -2 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 3 & 1 \end{pmatrix}$$

(c)
$$3\underset{\sim}{D} = 3 \times \begin{pmatrix} 5 & 7 \\ -4 & 2 \\ 5 & 3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 15 & 21 \\ -12 & 6 \\ 15 & 9 \\ -3 & 0 \end{pmatrix}$$

(d)
$$\underset{\sim}{A} + \underset{\sim}{D} = \begin{pmatrix} 1 & 3 \\ 4 & 2 \\ 0 & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ -4 & 2 \\ 5 & 3 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 0 & 4 \\ 5 & 2 \\ 0 & 0 \end{pmatrix}$$

(e)
$$\underset{\sim}{E}\,\underset{\sim}{B} = \begin{pmatrix} 2 & 1 & -1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 0 & -1 \end{pmatrix}$$

(f)
$$\underset{\sim}{I_3}\,\underset{\sim}{B} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

(g) $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \underset{\sim}{B}$

$$= \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ -2 & 1 \\ 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 3 \end{pmatrix}$$

$\therefore \quad \begin{pmatrix}1 & 1 & 1\end{pmatrix} \underset{\sim}{B} \begin{pmatrix}1\\1\end{pmatrix}$

$= \begin{pmatrix}5 & 3\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}$

$= \begin{pmatrix}8\end{pmatrix}$

(4)

$$\underset{\sim}{D}^T \underset{\sim}{D} = \begin{pmatrix}1 & 1\\-1 & 1\\1 & -1\\-1 & -1\end{pmatrix}\begin{pmatrix}1 & -1 & 1 & -1\\1 & 1 & -1 & -1\end{pmatrix} = \begin{pmatrix}2 & 0 & 0 & -2\\0 & 2 & -2 & 0\\0 & -2 & 2 & 0\\-2 & 0 & 0 & 2\end{pmatrix}$$

$$\underset{\sim}{D}\underset{\sim}{D}^T = \begin{pmatrix}1 & -1 & 1 & -1\\1 & 1 & -1 & -1\end{pmatrix}\begin{pmatrix}1 & 1\\-1 & 1\\1 & -1\\-1 & -1\end{pmatrix} = \begin{pmatrix}4 & 0\\0 & 4\end{pmatrix}$$

(5) (a) $\underset{\sim}{A}^2$

$= \begin{pmatrix}2 & 4\\-3 & 1\end{pmatrix}\begin{pmatrix}2 & 4\\-3 & 1\end{pmatrix}$

$= \begin{pmatrix}-8 & 12\\-9 & -11\end{pmatrix}$

(b) $\underset{\sim}{C}^2$

$= \begin{pmatrix}1 & -2\\7 & 0\end{pmatrix}\begin{pmatrix}1 & -2\\7 & 0\end{pmatrix}$

$= \begin{pmatrix}-13 & -2\\7 & -14\end{pmatrix}$

$\underset{\sim}{C}^3$

$= \underset{\sim}{C} \times \underset{\sim}{C}^2$

$= \begin{pmatrix}1 & -2\\7 & 0\end{pmatrix}\begin{pmatrix}-13 & -2\\7 & -14\end{pmatrix}$

$= \begin{pmatrix}-27 & 26\\-91 & -14\end{pmatrix}$

(c)

$$4\underset{\sim}{A} + 3\underset{\sim}{X} = \underset{\sim}{B} + 2\underset{\sim}{C}^T - 2\underset{\sim}{X}$$

$$5\underset{\sim}{X} = \underset{\sim}{B} + 2\underset{\sim}{C}^T - 4\underset{\sim}{A}$$

$$5\underset{\sim}{X} = \begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} + 2\begin{pmatrix} 1 & 7 \\ -2 & 0 \end{pmatrix} - 4\begin{pmatrix} 2 & 4 \\ -3 & 1 \end{pmatrix}$$

$$5\underset{\sim}{X} = \begin{pmatrix} -2 & 5 \\ 6 & -1 \end{pmatrix} + \begin{pmatrix} 2 & 14 \\ -4 & 0 \end{pmatrix} - \begin{pmatrix} 8 & 16 \\ -12 & 4 \end{pmatrix}$$

$$5\underset{\sim}{X} = \begin{pmatrix} -8 & 3 \\ 14 & -5 \end{pmatrix}$$

$$\underset{\sim}{X} = \frac{1}{5}\begin{pmatrix} -8 & 3 \\ 14 & -5 \end{pmatrix}$$

$$\underset{\sim}{X} = \begin{pmatrix} -8/5 & 3/5 \\ \frac{14}{5} & -1 \end{pmatrix}$$

(7)

$$\underset{\sim}{P}\,\underset{\sim}{Q}$$

$$= \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} c & d \\ -d & c \end{pmatrix}$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -ad\ \ bc & ac-bd \end{pmatrix}$$

$$\underset{\sim}{Q}\,\underset{\sim}{P}$$

$$= \begin{pmatrix} c & d \\ -d & c \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$$= \begin{pmatrix} ac-bd & ad+bc \\ -ad-bc & ac-bd \end{pmatrix}$$

$$\therefore\ \underset{\sim}{P}\,\underset{\sim}{Q} = \underset{\sim}{Q}\,\underset{\sim}{P}$$

(8) $\underset{\sim}{A}\,\underset{\sim}{B}$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$$

$\therefore (\underset{\sim}{A}\,\underset{\sim}{B})^T$

$$= \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$$

$\underset{\sim}{B}^T\,\underset{\sim}{A}^T$

$$= \begin{pmatrix} e & g \\ f & h \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix}$$

$$= \begin{pmatrix} ae+bg & ce+dg \\ af+bh & cf+dh \end{pmatrix}$$

$$\therefore (\underset{\sim}{A}\,\underset{\sim}{B})^T = \underset{\sim}{B}^T\,\underset{\sim}{A}^T$$

(10) (a)

$$(1-h)\times 1.4 = 1$$
$$1-h = \frac{1}{1.4}$$
$$h = 1-\frac{1}{1.4}$$
$$h \approx 0.286$$

A culling factor of 28.6% pa would be needed to achieve a stable population level.

(b)

$$(1-h)\times 1.4 = 1.15$$
$$1-h = \frac{1.15}{1.4}$$
$$h = 1-\frac{1.15}{1.4}$$
$$h \approx 0.179$$

A culling factor of 17.9% pa would be needed to achieve a long-term

(11). (a) The Leslie matrix is :

$$\underset{\sim}{L} = \begin{pmatrix} 0 & 0.4 & 1.5 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.9 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

(b) The initial female population is given by :

$$\underset{\sim}{X}_0 = \begin{pmatrix} 1800 \\ 1500 \\ 1150 \\ 700 \\ 400 \end{pmatrix}$$

Assuming that there are equal numbers of males and females, the total population in 6 years is given by :

$$(2 \quad 2 \quad 2 \quad 2 \quad 2)\, \underset{\sim}{L}^3\, \underset{\sim}{X}_0 = (17090)$$

The total population is estimated to be 17090 in 6 years.

(c) The revised Leslie matrix is :

$$\underset{\sim}{L}_1 = 0.8\, \underset{\sim}{L}$$

$$(2 \quad 2 \quad 2 \quad 2 \quad 2)\, \underset{\sim}{L}_1^5\, \underset{\sim}{X}_0 = (7743)$$

If 20% of the population is harvested every 2 years, the total population is estimated to be 7743 in 10 years.

(d) The revised Leslie matrix is:

$$\underset{\sim}{L_2} = \begin{pmatrix} 0 & 0.4\times0.7 & 1.5 & 1.2 & 0.3 \\ 0.6 & 0 & 0 & 0 & 0 \\ 0 & 0.9\times0.7 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0 \end{pmatrix}$$

$$(2 \quad 2 \quad 2 \quad 2 \quad 2)\ \underset{\sim}{L_2}^{5}\ \underset{\sim}{X_0} = (13741)$$

If 30% of 1-2 year olds are harvested each year, the total population is estimated to be 13741 in 10 years.

(e)

$$(2 \quad 2 \quad 2 \quad 2 \quad 2)\ \underset{\sim}{L}^{20}\ \underset{\sim}{X_0} = (190462)$$

$$(2 \quad 2 \quad 2 \quad 2 \quad 2)\ \underset{\sim}{L}^{21}\ \underset{\sim}{X_1} = (219178)$$

$$\frac{219178}{190462} \simeq 1.151$$

∴ If no harvesting takes place, the long-term growth rate is approximately 15.1% every 2 years.