MODELLING POPULATIONS BY LESLIE MATRICES

1) A population of bats living in a cave is studied and the following data is collected.

Age (months)	0 - 6	6 - 12	12 - 18	18 - 24
Initial population	4500	1800	900	130
Birth rate	0	1·9	1·5	0·7
Death rate	0·5	0·2	0·6	1

(a) Calculate the survival rates.
(b) What is the total population after 6, 12, 18 months and after 5 years?
(c) In the long term, what is the % increase in the population every 6 months?

2) A population of kangaroos is studied and the following data is collected.

Age (years)	0 - 2	2 - 4	4 - 6	6 - 8	8 - 10
Initial population	3400	2500	2300	1750	650
Breeding rate	0	0	3·9	2·7	0·9
Survival rate	0·5	0·8	0·7	0·4	0

Suggest a suitable culling rate to maintain a stable population if the culling is carried out every 2 years.

3) A population of buffalo in a certain area is farmed as a resource. The following data is available.

Age (years)	0 - 1	1 - 2	2 - 3	3 - 4	4 - 5	6 - 7
Initial population	2350	2000	2000	1450	825	400
Breeding rate	0	0	1·5	1·7	1·2	0·4
Survival rate	0·6	0·7	0·9	0·5	0·3	0

Investigate the effect on the population of different harvesting rates and prepare a report. Only buffalo aged 2 - 4 years would be slaughtered for the meat and the skins.

(1) (a) The survival rates are "1 − death rate" :-

age 0 − 6 months		$1 - 0.5 = 0.5$
" 6 − 12	"	$1 - 0.2 = 0.8$
" 12 − 18	"	$1 - 0.6 = 0.4$
" 18 − 24	"	$1 - 1 = 0$

(b) The Leslie matrix is :

$$\underset{\sim}{L} = \begin{pmatrix} 0 & 1.9 & 1.5 & 0.7 \\ 0.5 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 \\ 0 & 0 & 0.4 & 0 \end{pmatrix}$$

The initial population is given by :

$$\underset{\sim}{X}_0 = \begin{pmatrix} 4500 \\ 1800 \\ 900 \\ 130 \end{pmatrix}$$

The total population after 6 months is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L} \underset{\sim}{X}_0 = (8911)$$

The total population after 12 months is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L}^2 \underset{\sim}{X}_0 = (11494)$$

The total population after 18 months is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L}^3 \underset{\sim}{X}_0 = (13729)$$

The total population after 5 years is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L}^{10} \underset{\sim}{X}_0 = (58890)$$

Consider the total population after:

5 years $\quad (1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{10} \, \underset{\sim}{X}_0 = (58890)$

$5\frac{1}{2}$ years $\quad (1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{11} \, \underset{\sim}{X}_0 = (72415)$

6 years $\quad (1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{12} \, \underset{\sim}{X}_0 = (89044)$

$$\frac{72415}{58890} \approx 1.230 \quad \text{and} \quad \frac{89044}{72415} \approx 1.230$$

$\therefore$ In the long term, the population increases by 23.0% every 6 months.

(2) The Leslie matrix is:

$$\underset{\sim}{L} = \begin{pmatrix} 0 & 0 & 3.9 & 2.7 & 0.9 \\ 0.5 & 0 & 0 & 0 & 0 \\ 0 & 0.8 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 \end{pmatrix}$$

The initial population is given by:

$$\underset{\sim}{X}_0 = \begin{pmatrix} 3400 \\ 2500 \\ 2300 \\ 1750 \\ 650 \end{pmatrix}$$

Consider the total population after:

20 years $\quad (1 \; 1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{10} \, \underset{\sim}{X}_0 = (175313)$

22 years $\quad (1 \; 1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{11} \, \underset{\sim}{X}_0 = (246427)$

24 years $\quad (1 \; 1 \; 1 \; 1 \; 1) \, \underset{\sim}{L}^{12} \, \underset{\sim}{X}_0 = (299698)$

$$\frac{246427}{175313} \approx 1.406 \qquad \text{and} \qquad \frac{299698}{246427} \approx 1.216$$

These growth factors suggest that the population growth is erratic using the data provided and the Leslie matrix model.

Assuming a growth rate of 31% every two years:

$$(1-h) \times 1.31 = 1$$

$$1-h = \frac{1}{1.31}$$

$$h = 1 - \frac{1}{1.31}$$

$$h \approx 0.24$$

$\therefore$ The culling rate would be 24%.

(3) Suppose the harvesting rate is h for 2 - 4 year old buffalo.

The Leslie matrix is:

$$\underset{\sim}{L} = \begin{pmatrix} 0 & 0 & 1.5(1-h) & 1.7(1-h) & 1.2 & 0.4 \\ 0.6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.9(1-h) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5(1-h) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 \end{pmatrix}$$

The initial population is given by:

$$\underset{\sim}{X}_0 = \begin{pmatrix} 2350 \\ 2000 \\ 2000 \\ 1450 \\ 825 \\ 400 \end{pmatrix}$$

The total population after n years is given by:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \underset{\sim}{L}^n \underset{\sim}{X}_0$$

Consider different values for h :

h	total population $n=10$	$n=11$	$n=12$
0	30628	34021	38688
0·1	19318	20615	22365
0·2	11617	11842	12196
0·21	11007	11165	11435
0·22	10423	10520	10714
0·23	9864	9905	10031
0·24	9329	9320	9385
0·25	8817	8763	8772

A harvesting rate of approximately 25%
would result in a stable population.