LESLIE MATRICES

Leslie matrices are used to *model* the changes in a population.

The technique concentrates on the *female population* as it is females which reproduce.

The population needs to be considered in age groups. The age groups are best chosen to correspond to the *length of a generation* ie. the number of years before a new born female reaches fertile age.

We need to know the following information for the female population:
(a) the *initial population* in each age group
(b) the *breeding rate* of each age group
(c) the *survival rate* of each age group.

Consider the following example:

DETAILS FOR FEMALE POSSUM POPULATION					
Age (years)	0 - 1	1 - 2	2 - 3	3 - 4	4 - 5
Initial population	194	82	55	22	6
Breeding rate	0	1·3	1·8	0·9	0·2
Survival rate	0·6	0·8	0·8	0·4	0

Notes:
(a) There are initially 82 females who are between their 1st and 2nd birthday.
(b) It is expected that 1·3 female offspring will be produced during the year and will survive to the end of the year for each of the 82 females alive at the beginning of the year.
(c) The proportion of the 82 females expected to survive one year is 0·8 .

The initial population can be represented by the column matrix $\mathbf{X}_0 = \begin{pmatrix} 194 \\ 82 \\ 55 \\ 22 \\ 6 \end{pmatrix}$.

The population after 1 year is given by $\mathbf{X}_1 = \mathbf{L}\,\mathbf{X}_0$ where $\mathbf{L}$ is the Leslie matrix.

$$\mathbf{L} = \begin{pmatrix} 0 & 1\cdot 3 & 1\cdot 8 & 0\cdot 9 & 0\cdot 2 \\ 0\cdot 6 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot 8 & 0 & 0 & 0 \\ 0 & 0 & 0\cdot 8 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot 4 & 0 \end{pmatrix}$$

This is explained as follows:

$$\mathbf{X}_1$$

$$= \begin{pmatrix} \text{births during year} \\ \text{survivors of } 0\text{-}1 \text{ population} \\ \text{survivors of } 1\text{-}2 \text{ population} \\ \text{survivors of } 2\text{-}3 \text{ population} \\ \text{survivors of } 3\text{-}4 \text{ population} \end{pmatrix}$$

$$= \begin{pmatrix} 0\times194+1\cdot3\times82+1\cdot8\times55+0\cdot9\times22+0\cdot2\times6 \\ 0\cdot6\times194 \\ 0\cdot8\times82 \\ 0\cdot8\times55 \\ 0\cdot4\times22 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1\cdot3 & 1\cdot8 & 0\cdot9 & 0\cdot2 \\ 0\cdot6 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot8 & 0 & 0 & 0 \\ 0 & 0 & 0\cdot8 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot4 & 0 \end{pmatrix} \begin{pmatrix} 194 \\ 82 \\ 55 \\ 22 \\ 6 \end{pmatrix}$$

$$= \begin{pmatrix} 226\cdot6 \\ 116\cdot4 \\ 65\cdot6 \\ 44\cdot0 \\ 8\cdot8 \end{pmatrix}$$

Repeated multiplication by the Leslie matrix gives the female population in 2, 3, ... years.

$$\mathbf{X}_2 = \mathbf{L}\mathbf{X}_1 = \begin{pmatrix} 0 & 1\cdot3 & 1\cdot8 & 0\cdot9 & 0\cdot2 \\ 0\cdot6 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot8 & 0 & 0 & 0 \\ 0 & 0 & 0\cdot8 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot4 & 0 \end{pmatrix} \begin{pmatrix} 226\cdot6 \\ 116\cdot4 \\ 65\cdot6 \\ 44\cdot0 \\ 8\cdot8 \end{pmatrix} = \begin{pmatrix} 310\cdot8 \\ 136\cdot0 \\ 93\cdot1 \\ 52\cdot5 \\ 17\cdot6 \end{pmatrix}$$

$$\mathbf{X}_3 = \mathbf{L}\mathbf{X}_2 = \begin{pmatrix} 0 & 1\cdot3 & 1\cdot8 & 0\cdot9 & 0\cdot2 \\ 0\cdot6 & 0 & 0 & 0 & 0 \\ 0 & 0\cdot8 & 0 & 0 & 0 \\ 0 & 0 & 0\cdot8 & 0 & 0 \\ 0 & 0 & 0 & 0\cdot4 & 0 \end{pmatrix} \begin{pmatrix} 310\cdot8 \\ 136\cdot0 \\ 93\cdot1 \\ 52\cdot5 \\ 17\cdot6 \end{pmatrix} = \begin{pmatrix} 395\cdot2 \\ 186\cdot5 \\ 108\cdot8 \\ 74\cdot5 \\ 21\cdot0 \end{pmatrix} \text{ etc.}$$

Another way of looking at this process is:

$$\mathbf{X}_1 = \mathbf{L}\,\mathbf{X}_0$$
$$\mathbf{X}_2 = \mathbf{L}^2\,\mathbf{X}_0$$
$$\mathbf{X}_3 = \mathbf{L}^3\,\mathbf{X}_0$$
$$\ldots$$
$$\mathbf{X}_n = \mathbf{L}^n\,\mathbf{X}_0$$

The *Leslie matrix model of a population leads to stable population growth and stable age distribution*. This is shown in the following table of age distributions for different generations. It has been assumed that *half of the total population is female*.

Generation	10	11	12	13	14
0 - 1	2378·3	3073·5	3971·5	5132·1	6631·8
1 - 2	1104·3	1427·0	1844·1	2382·9	3079·2
2 - 3	683·8	883·5	1141·6	1475·3	1906·3
3 - 4	423·1	547·1	706·8	913·3	1180·2
4 - 5	131·0	169·2	218·8	282·7	365·3
Total female population	4720·5	6100·3	7882·8	10186·3	13162·8
Total population	9441·0	12200·6	15765.6	20372·6	26325·6

The total population is increasing by a constant rate of 29·2% pa (ie. $\mathbf{X}_n = k\,\mathbf{X}_{n-1}$ where k = 1·292) and the age distribution has stabilised as follows.

Age (years)	0 - 1	1 - 2	2 - 3	3 - 4	4 - 5
% of population	50·4%	23·4%	14·5%	9·0%	2·8%

The model can be adjusted to allow for *culling* or *harvesting*. Suppose that, *for each age group*, a proportion h = 0·15 or 15% of the possum population is culled at the beginning of the year. This is equivalent to using a new Leslie matrix $\mathbf{L}' = (1-h)\mathbf{L} = 0\cdot 85\,\mathbf{L}$.

To achieve a *stable population level*, the culling factor h is chosen so that:

$$(1-h)k = 1$$
$$(1-h)\times 1\cdot 292 = 1$$
$$1-h = \frac{1}{1\cdot 292}$$
$$h = 1 - \frac{1}{1\cdot 292}$$
$$h = 0\cdot 226$$