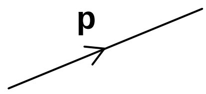
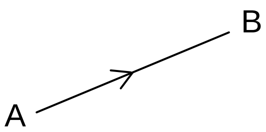
GEOMETRIC VECTORS

VECTOR NOTATION

A *geometric vector* is a quantity which has *magnitude* and *direction*. Examples of geometric vectors are displacement, translation, velocity, acceleration, force, momentum, impulse.

The magnitude (sometimes called *norm*) of vector **p** is denoted by $|\mathbf{p}|$.

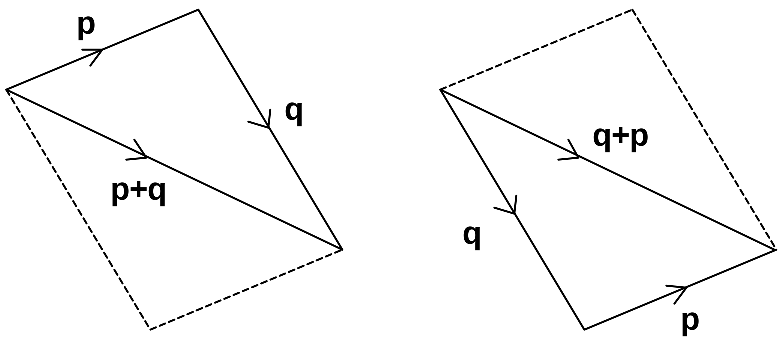




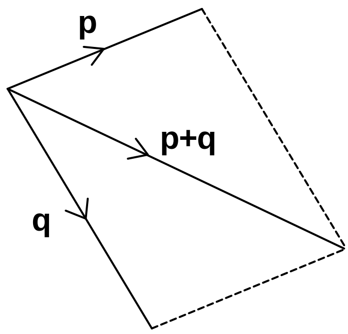
A displacement from point A to point B is denoted by the vector **AB**. The magnitude of **AB** is written as $|\mathbf{AB}|$ or AB.

VECTOR ADDITION

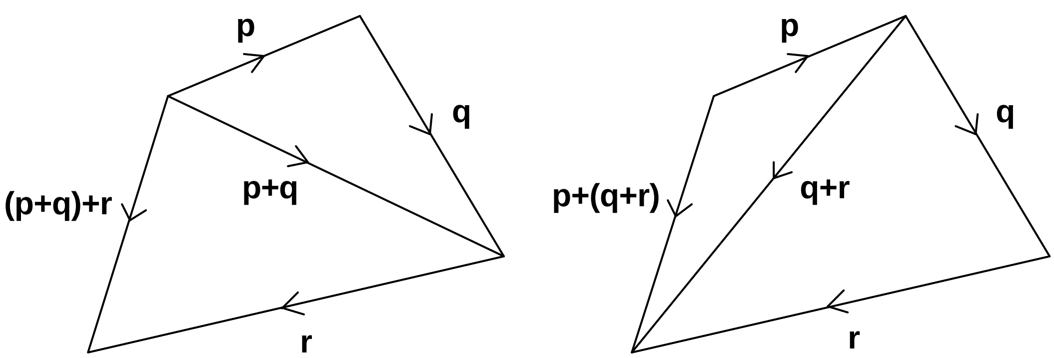
Two geometric vectors are added by the *triangle law of addition* ie. *head-to-tail*. (See previous work on the addition of complex numbers treated as displacements.) The addition of geometric vectors is *commutative* ie. $\mathbf{p}+\mathbf{q}=\mathbf{q}+\mathbf{p}$.



The commutativity of addition leads to the *alternative parallelogram law of addition* ie. if two vectors are represented by adjacent sides of a parallelogram, then their sum is represented by the diagonal through the point of intersection of the two sides.



Addition of geometric vectors is *associative* ie. $(\mathbf{p}+\mathbf{q})+\mathbf{r}=\mathbf{p}+(\mathbf{q}+\mathbf{r})$.

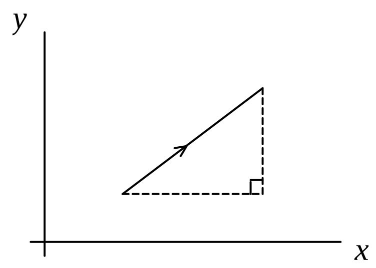


The triangle law of addition can be extended so that any number of vectors can be added by the *head-to tail* method.

VECTORS IN TWO DIMENSIONS (CARTESIAN FORM)

Vectors in two dimensions can be represented in cartesian form.

Consider the vector, eg. a displacement, which is equivalent to 4 in the x-direction and 3 in the y-direction. As with storage vectors (see previous topic), this vector can be represented as an ordered pair, row matrix or column matrix:



$$(4,3) \quad \text{or} \quad \begin{pmatrix} 4 & 3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4 \\ 3 \end{pmatrix}$$

Other notation is available by considering the vector as the sum of a vector of magnitude 4 in the x-direction and a vector of magnitude 3 in the y-direction:

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 3 \end{pmatrix} = 4\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is the *unit vector* (ie. magnitude 1) in the x-direction and can be denoted by $\mathbf{i}$.

The vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is the *unit vector* (ie. magnitude 1) in the y-direction and can be denoted by $\mathbf{j}$

The cartesian form can therefore also be written as $4\mathbf{i} + 3\mathbf{j}$.

Pythagoras' theorem gives the magnitude: $|4\mathbf{i} + 3\mathbf{j}| = \sqrt{4^2 + 3^2} = 5$

In general: $|x\mathbf{i} + y\mathbf{j}| = \sqrt{x^2 + y^2}$

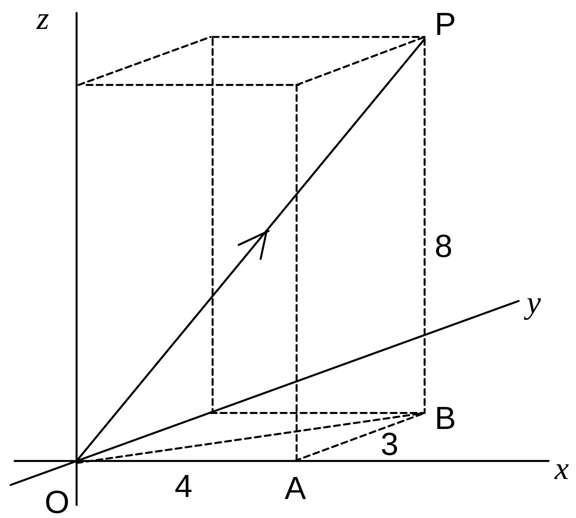
VECTORS IN THREE DIMENSIONS (CARTESIAN FORM)

Vectors in three dimensions can be represented in cartesian form.

Consider the vector which is equivalent to 4 in the x-direction, 3 in the y-direction and 8 in the z-direction. This vector can be represented as:

$$(4,3,8) \quad \text{or} \quad \begin{pmatrix} 4 & 3 & 8 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 4 \\ 3 \\ 8 \end{pmatrix} \quad \text{or} \quad 4\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}$$

where $\mathbf{k}$ is the unit vector in the z-direction



Pythagoras' theorem (applied twice) gives the magnitude:

$$|4\mathbf{i} + 3\mathbf{j} + 8\mathbf{k}| = \sqrt{(\text{OB})^2 + (\text{BP})^2} = \sqrt{\left((\text{OA})^2 + (\text{AB})^2\right) + (\text{BP})^2} = \sqrt{4^2 + 3^2 + 8^2} = \sqrt{89}$$

In general: $|x\mathbf{i} + y\mathbf{j} + z\mathbf{k}| = \sqrt{x^2 + y^2 + z^2}$

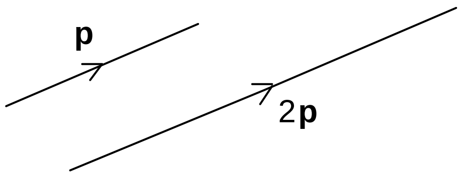
ZERO VECTOR

The zero vector $\mathbf{0}$ has zero magnitude but it has no direction. In cartesian form:

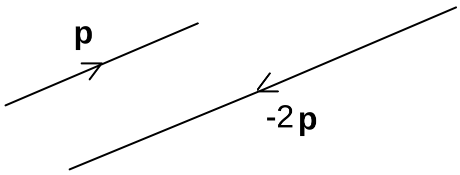
$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ in two dimensions

$\mathbf{0} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ in three dimensions

MULTIPLICATION OF A VECTOR BY A SCALAR



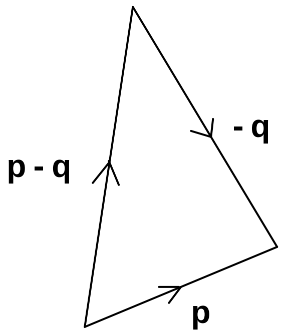
When a vector **p** is multiplied by a scalar $n > 0$, then the vector $n\mathbf{p}$ has the same direction as **p** and its magnitude is multiplied by n.



When a vector **p** is multiplied by a scalar $n < 0$, then the vector $n\mathbf{p}$ has the opposite direction to **p** and its magnitude is multiplied by $|n|$.

SUBTRACTION OF VECTORS

A vector is subtracted by adding its negative.



MATRIX ARITHMETIC & VECTORS AS A GROUP

Matrix arithmetic is wholly compatible with the definitions for the addition of vectors, the multiplication of a vector by a scalar and the subtraction of a vector. Eg. The calculations opposite can be checked against the definitions.

$$\begin{pmatrix} 4 \\ 3 \end{pmatrix} + \begin{pmatrix} 6 \\ 2 \end{pmatrix} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$$

$$2\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 8 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 10 \\ 5 \end{pmatrix} - \begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ 2 \end{pmatrix}$$

Also two dimensional vectors form a group under addition (as do three dimensional vectors) ie. they satisfy the closure, associative, identity and inverse properties.

UNIT VECTORS

Unit vectors in the direction of a given vector can be found by dividing by its magnitude.
Eg. If $\mathbf{p} = 4\mathbf{i} + 3\mathbf{j}$, then the unit vector in the direction of **p** is denoted by $\hat{\mathbf{p}}$ where $\hat{\mathbf{p}} = \frac{1}{5}(4\mathbf{i} + 3\mathbf{j})$.

RESULTANT & COMPONENTS

The sum of a number of vectors is called the *resultant*. The vectors in the sum are called the *components*. Eg. If $\mathbf{p} = \mathbf{a} + \mathbf{b} + \mathbf{c}$, then $\mathbf{p}$ is the resultant of $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are the components of $\mathbf{p}$. It is usual to denote a resultant vector on a diagram by a double arrow.

RESULTANT OF FORCES ACTING AT A POINT - GEOMETRIC METHODS

Problems involving the resultant of forces acting at a point can be solved by drawing head-to-tail diagrams (or using the parallelogram law of addition) and solving the resulting diagram by geometric methods eg. trigonometry, cosine rule, sine rule.

Eg. Forces 12 newtons and 8 newtons acting at a point have a resultant of 9 newtons. What is the angle between the forces?

Consider the triangle law of addition.

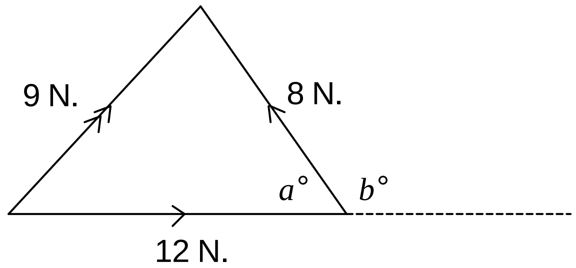
by cosine rule:

$$\cos a^\circ = \frac{12^2 + 8^2 - 9^2}{2 \times 12 \times 8}$$

$$a = 48 \cdot 6$$

$$b = 180 - 48 \cdot 6$$

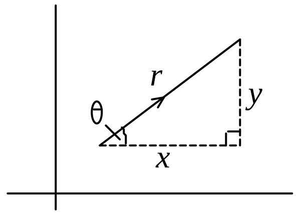
$$b = 131 \cdot 4$$



$\therefore$ Angle between forces is 131·4°.

VECTORS IN TWO DIMENSIONS (POLAR FORM)

Two dimensional vectors in cartesian form $x\mathbf{i} + y\mathbf{j}$ can be represented in *polar form* (r, q) where r is the magnitude and q is the angle made by the vector with the positive direction of the x-axis.



From trigonometry, $x = r\cos q$ and $y = r\sin q$.

To switch between cartesian and polar form, use the special functions on the calculator or base the calculations on a diagram.

RESULTANT OF FORCES ACTING AT A POINT - USE OF POLAR FORM

Forces can be expressed in polar form, added in cartesian form and the resultant converted to polar form.

Eg. Three forces act a point - 13 newtons acting N30°E, 18 newtons acting S40°W and 15 newtons acting west. What is the resultant?

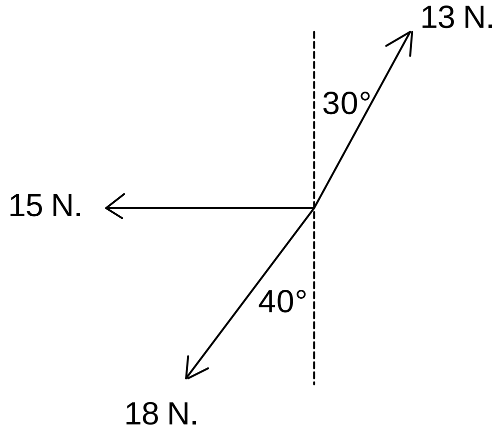
resultant

$$= (13, 60^\circ) + (15, 180^\circ) + (18, 230^\circ)$$

$$= \begin{pmatrix} 13\cos 60^\circ \\ 13\sin 60^\circ \end{pmatrix} + \begin{pmatrix} 15\cos 180^\circ \\ 15\sin 180^\circ \end{pmatrix} + \begin{pmatrix} 18\cos 230^\circ \\ 18\sin 230^\circ \end{pmatrix}$$

$$= \begin{pmatrix} {}^{-}20 \cdot 07017697 \\ {}^{-}2 \cdot 530469727 \end{pmatrix}$$

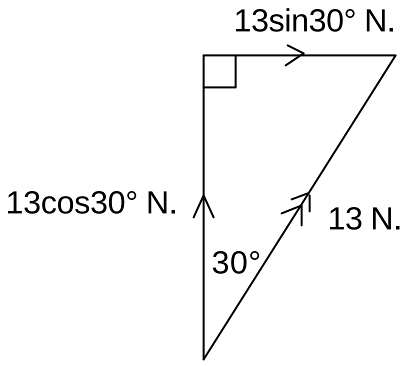
$$= (20 \cdot 2,\ {}^{-}172 \cdot 8^\circ)$$



∴ The resultant is 20·2 newtons acting on a bearing of 262·8°.

REPLACING A VECTOR BY COMPONENTS ACTING AT RIGHT ANGLES

Consider a force of 13 newtons acting in the direction N30°E and suppose that the force is to be replaced by two separate forces acting north and east. The 13 Newton force is the sum or resultant of the two separate forces. Consider the triangle law of addition.



By trigonometry, the *components* are 13cos30° and 13sin30°. The important *pattern* is that the “cosine” component is in the direction which makes the given angle with the vector being replaced by components.

FORCES ACTING AT A POINT IN EQUILIBRIUM - TAKING COMPONENTS IN A CERTAIN DIRECTION

When forces act at a point and have a resultant of zero, then those forces are said to be in *equilibrium*. A clear diagram showing all the forces acting should first be drawn. A consideration of components acting in a certain direction provides an equation. A consideration of components acting in a second direction provides a second equation.

Eg. A 2 kg particle is suspended by a string. A horizontal force holds the particle in equilibrium with the string making an angle of 25° with the vertical. What is the size of the force?

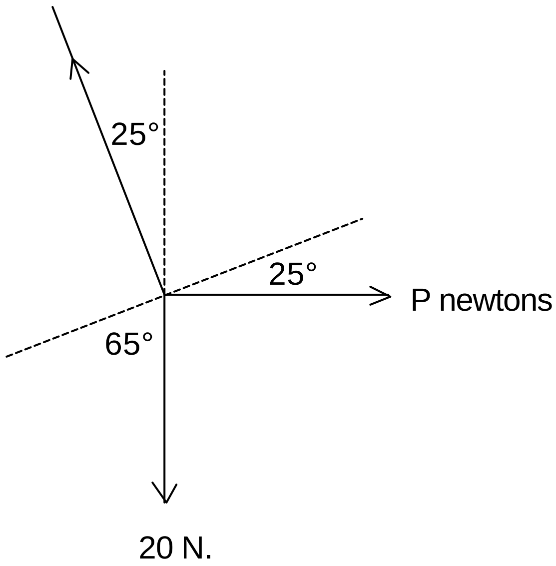
R(perpendicular to the string):

$$P\cos 25^\circ = 20\cos 65^\circ$$

$$P = \frac{20\cos 65^\circ}{\cos 25^\circ}$$

$$P = 9 \cdot 3$$

The force is 9·3 N.



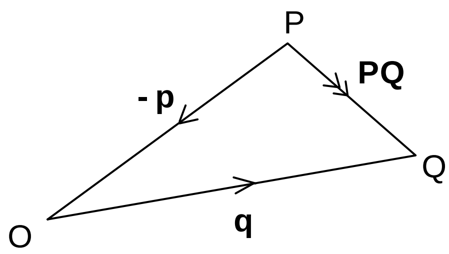
VECTOR METHODS IN GEOMETRY

Position vectors and notation

A point O is called the *origin* if it is used to fix the positions of points. The vector **OP** is called the *position vector* of the point P. The position vector **OP** is often denoted by $\mathbf{p}$.

Displacement vector as the difference between two position vectors

Consider points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$ with respect to some origin O.



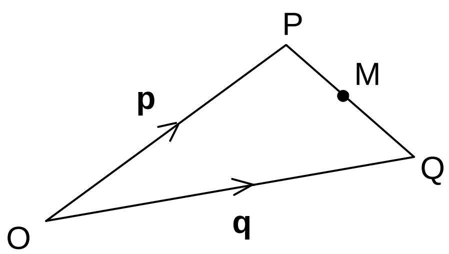
$$\begin{aligned}&\mathbf{PQ}\\ &= (-\mathbf{OP}) + \mathbf{OQ}\\ &= \mathbf{OQ} - \mathbf{OP}\\ &= \mathbf{q} - \mathbf{p}\end{aligned}$$

This back to front subtraction pattern applies to any point not just O eg. $\mathbf{PQ} = \mathbf{AQ} - \mathbf{AP}$.

Position vector of midpoint of line

Consider points P and Q with position vectors $\mathbf{p}$ and $\mathbf{q}$ with respect to some origin O.

Let M be the midpoint of PQ.



$$\begin{aligned}&\mathbf{OM}\\ &= \mathbf{OP} + \mathbf{PM}\\ &= \mathbf{OP} + \tfrac{1}{2}\mathbf{PQ}\\ &= \mathbf{p} + \tfrac{1}{2}(\mathbf{q} - \mathbf{p})\\ &= \mathbf{p} + \tfrac{1}{2}\mathbf{q} - \tfrac{1}{2}\mathbf{p}\\ &= \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}\\ &= \tfrac{1}{2}(\mathbf{p} + \mathbf{q})\end{aligned}$$

This average pattern applies to any point not just O eg. $\mathbf{AM} = \tfrac{1}{2}(\mathbf{AP} + \mathbf{AQ})$.