GEOMETRIC VECTORS REVIEW

1. A geometric vector is a quantity which has ____ and ____ ?

2. If P is the point $(5,-2)$ and Q is the point $(-3,7)$, evaluate **PQ** and PQ?

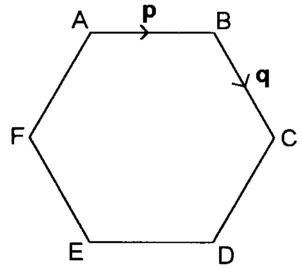
3. If P is the point $(5,3,-2)$ and Q is the point $(-3,7,2)$, evaluate **PQ** and PQ?

4. The position vectors of points A and B with respect to an origin are **a** and **b**. Express **AB**, **BA** and the position vector of the midpoint of AB in terms of **a** and **b**.

5. Consider the points $A(3,-2,1)$, $B(-6,5,-1)$ and $C(2,4,-3)$.
   a) Find **BC** and $|\mathbf{CA}|$.
   b) M is the midpoint of AC and N is the midpoint of BM. What is the position vector of N?
   c) If D is on the line AB produced so that AD = 3 x AB, what is the position vector of D?

6. On the cartesian plane, ABCD is a rectangle such that A is the point (5,-2), $\mathbf{AB} = 3\mathbf{i} + 4\mathbf{j}$ and $\mathbf{AD} = -4\mathbf{i} + 3\mathbf{j}$. Find the coordinates of C.

7. ABCDEF is a regular hexagon. $\mathbf{AB} = \mathbf{p}$ and $\mathbf{BC} = \mathbf{q}$. Express in terms of **p** and **q** the vectors **AC**, **AD**, **EA** and **CF**.



8. If $\mathbf{p} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$, $\mathbf{q} = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, evaluate $2\mathbf{p} - 3\mathbf{q} + \mathbf{r}$ and $|\mathbf{p} + \mathbf{q}|$.

9. If $\mathbf{p} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$, $\mathbf{q} = 2\mathbf{i} - 3\mathbf{k}$ and $\mathbf{r} = 5\mathbf{j} + \mathbf{k}$, evaluate $\mathbf{p} + 2\mathbf{q} - 3\mathbf{r}$ and $|\mathbf{r} - \mathbf{q}|$.

10. The vector $\begin{pmatrix} 8 \\ -3 \\ a \end{pmatrix}$ is parallel to $\begin{pmatrix} b \\ 5 \\ -6 \end{pmatrix}$. Evaluate a and b.

11. If $|e\mathbf{i} - (e-1)\mathbf{j}| = 5$, find the value of e.

12. The vector $\begin{pmatrix} k+1 \\ 2k \\ k-3 \end{pmatrix}$ has magnitude $\sqrt{12}$. What is the value of k?

13. What is the unit vector $\hat{\mathbf{a}}$ in the direction of $\mathbf{a} = 2\mathbf{i} - 2\mathbf{j} - \mathbf{k}$?

14. Draw a diagram to demonstrate that vector addition is commutative ie. $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$.

15. Draw a diagram to demonstrate that vector addition is associative ie. $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$.

16. Vectors of the same size form an Abelian group under addition - true or false?

17. Explain what is meant by the triangle law of addition of two vectors and the parallelogram law of addition of two vectors.

18. Forces of 7 N and 5 N act a point and have a resultant of 10 N. What is the angle between the two forces?

19. Two forces act at a point. The angle between the two forces is 60° and the magnitude of the resultant is 10 N. If one force has twice the magnitude of the other, find the two forces.

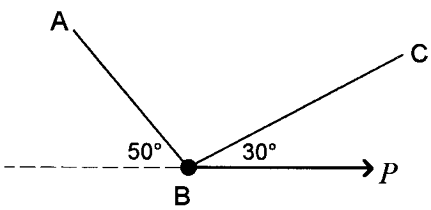
20. Find the resultant of the following forces acting at a point: 12 N acting NE, 15 N acting N75°E, 18 N acting S and 14 N acting on a bearing of 230°.

21. Three forces act at a point: 40 N on a bearing of 060°, 80 N on a bearing of 160° and a third force on a bearing of 290°. The resultant of the three forces acts south. What is the magnitude of the third force?

22. A 8 kg particle lies on a smooth plane inclined at an angle of 35° to the horizontal. A force of P newtons holds the particle in equilibrium. Find the value of P and the normal reaction if P acts:
a) horizontally
b) straight up the plane
c) up the plane at an angle of 20° with the line of greatest slope.

23. A 5 kg particle is suspended by a string. A force of P newtons acting on the particle at an angle of 15° above the horizontal causes it to rest in equilibrium with the string making an angle of 20° with the vertical. Find the tension in the string and the value of P.

24. A particle of mass 4 kg is suspended by two strings AB and BC which make angles 50° and 30° with the horizontal. A horizontal force of P newtons is applied to the particle so that it is held in equilibrium with the tension in AB twice the tension in BC. Calculate the value of P.



ANSWERS

(2) $\begin{pmatrix}-8\\9\end{pmatrix}$, $\sqrt{145}$ (3) $\begin{pmatrix}-8\\4\\4\end{pmatrix}$, $4\sqrt{6}$ (4) $\mathbf{b}-\mathbf{a}$, $\mathbf{a}-\mathbf{b}$, $\frac{1}{2}(\mathbf{a}+\mathbf{b})$

(5)(a) $8\mathbf{i}-\mathbf{j}-2\mathbf{k}$, $\sqrt{53}$ (b) $^{-}\frac{7}{4}\mathbf{i}+3\mathbf{j}-\mathbf{k}$ (c) $^{-}24\mathbf{i}+19\mathbf{j}-5\mathbf{k}$ (6) $(4,5)$

(7) $\mathbf{p}+\mathbf{q}$, $2\mathbf{q}$, $\mathbf{p}-2\mathbf{q}$, $-2\mathbf{p}$ (8) $\begin{pmatrix}19\\-19\end{pmatrix}$, $\sqrt{10}$ (9) $7\mathbf{i}-13\mathbf{j}-5\mathbf{k}$, $3\sqrt{5}$ (10) $\frac{18}{5}$, $-\frac{40}{3}$

(11) -3 or 4 (12) $-\frac{1}{3}$ or 1 (13) $\frac{2}{3}\mathbf{i}-\frac{2}{3}\mathbf{j}-\frac{1}{3}\mathbf{k}$ (18) 68.2° (19) 3.8 N, 7.6 N

(20) 19.1 N, 140.1° (21) 66.0 N (22) a) 56.0 N, 97.7 N b) 45.9 N, 65.5 N c) 48.8 N, 4.8N
(23) 48.5 N, 17.2 N (24) 8.3 N

(1) A geometric vector is a quantity which has magnitude and direction.

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(2) $\tilde{PQ}$

$= \tilde{OQ} - \tilde{OP}$

$= \begin{pmatrix} -3 \\ 7 \end{pmatrix} - \begin{pmatrix} 5 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -8 \\ 9 \end{pmatrix}$

PQ or $|\tilde{PQ}|$

$= \sqrt{[(-8)^2 + 9^2]}$

$= \sqrt{[64 + 81]}$

$= \sqrt{145}$

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(3) $\tilde{PQ}$

$= \tilde{OQ} - \tilde{OP}$

$= \begin{pmatrix} -3 \\ 7 \\ 2 \end{pmatrix} - \begin{pmatrix} 5 \\ 3 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -8 \\ 4 \\ 4 \end{pmatrix}$

PQ or $|\tilde{PQ}|$

$= \sqrt{[(-8)^2 + 4^2 + 4^2]}$

$= \sqrt{96}$

$= \sqrt{16 \times 6}$

$= 4\sqrt{6}$

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(4) $\tilde{AB}$

$= \tilde{OB} - \tilde{OA}$

$= \tilde{b} - \tilde{a}$

$\tilde{BA}$

$= \tilde{OA} - \tilde{OB}$

$= \tilde{a} - \tilde{b}$

position vector of midpoint

$= \frac{1}{2}(\tilde{a} + \tilde{b})$

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(5) (a) $\underset{\sim}{BC}$

$= \underset{\sim}{OC} - \underset{\sim}{OB}$

$= \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix} - \begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 8 \\ -1 \\ -2 \end{pmatrix}$

$\underset{\sim}{CA}$

$= \underset{\sim}{OA} - \underset{\sim}{OC}$

$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}$

$= \begin{pmatrix} 1 \\ -6 \\ 4 \end{pmatrix}$

$|\underset{\sim}{CA}|$

$= \sqrt{[1^2 + 6^2 + 4^2]}$

$= \sqrt{53}$

(b) $\underset{\sim}{OM}$

$= \frac{1}{2}(\underset{\sim}{OA} + \underset{\sim}{OC})$

$= \frac{1}{2}\left[\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \\ -3 \end{pmatrix}\right]$

$= \frac{1}{2}\begin{pmatrix} 5 \\ 2 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} 2.5 \\ 1 \\ -1 \end{pmatrix}$

$\underset{\sim}{ON}$

$= \frac{1}{2}(\underset{\sim}{OB} + \underset{\sim}{OM})$

$= \frac{1}{2}\left[\begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix} + \begin{pmatrix} 2.5 \\ 1 \\ -1 \end{pmatrix}\right]$

$= \frac{1}{2}\begin{pmatrix} -3.5 \\ 6 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -1.75 \\ 3 \\ -1 \end{pmatrix}$

$\therefore$ Position vector of N is $-1.75\underset{\sim}{i} + 3\underset{\sim}{j} - \underset{\sim}{k}$.

(c) $\underset{\sim}{AB}$

$= \underset{\sim}{OB} - \underset{\sim}{OA}$

$= \begin{pmatrix} -6 \\ 5 \\ -1 \end{pmatrix} - \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} -9 \\ 7 \\ -2 \end{pmatrix}$

$\underset{\sim}{AD}$

$= 3\underset{\sim}{AB}$

$= 3\begin{pmatrix} -9 \\ 7 \\ -2 \end{pmatrix}$

$= \begin{pmatrix} -27 \\ 21 \\ -6 \end{pmatrix}$

$\underset{\sim}{OD}$

$= \underset{\sim}{OA} + \underset{\sim}{AD}$

$= \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} -27 \\ 21 \\ -6 \end{pmatrix}$

$= \begin{pmatrix} -24 \\ 19 \\ -5 \end{pmatrix}$

$\therefore$ Position vector of D is $-24\underset{\sim}{i} + 19\underset{\sim}{j} - 5\underset{\sim}{k}$.

(6)

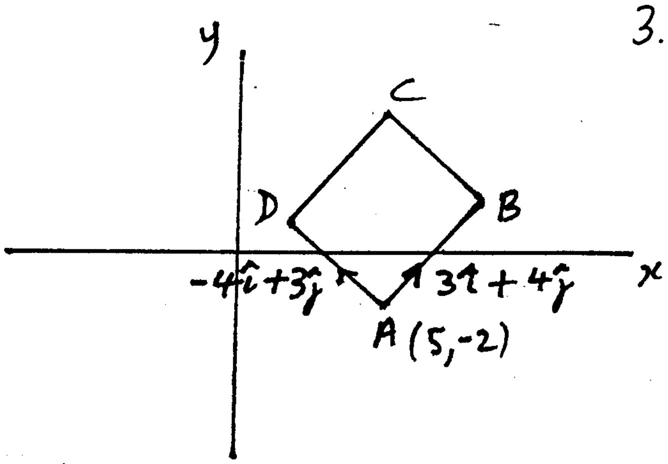
$\widetilde{OC}$

$= \widetilde{OA} + \widetilde{AB} + \widetilde{BC}$

$= \begin{pmatrix} 5 \\ -2 \end{pmatrix} + \begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} -4 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 4 \\ 5 \end{pmatrix}$

$\therefore$ C is the point $(4, 5)$



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(7)

$\widetilde{AC}$
$= \widetilde{AB} + \widetilde{BC}$
$= \tilde{p} + \tilde{q}$

$\widetilde{AD}$
$= 2\,\widetilde{BC}$
$= 2\tilde{q}$

$\widetilde{EA}$
$= \widetilde{ED} + \widetilde{DA}$
$= \tilde{p} + {}^{-}2\tilde{q}$
$= \tilde{p} - 2\tilde{q}$

$\widetilde{CF}$
$= 2\,\widetilde{BA}$
$= -2\tilde{p}$

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(8)

$2\tilde{p} - 3\tilde{q} + \tilde{r}$

$= 2\begin{pmatrix} 5 \\ -3 \end{pmatrix} - 3\begin{pmatrix} -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 10 \\ -6 \end{pmatrix} - \begin{pmatrix} -6 \\ 12 \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$= \begin{pmatrix} 19 \\ -19 \end{pmatrix}$

$|\tilde{p} + \tilde{q}|$

$= \left| \begin{pmatrix} 5 \\ -3 \end{pmatrix} + \begin{pmatrix} -2 \\ 4 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} 3 \\ 1 \end{pmatrix} \right|$

$= \sqrt{[3^2 + 1^2]}$

$= \sqrt{10}$

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(9)

$\tilde{p} + 2\tilde{q} - 3\tilde{r}$

$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} - 3\begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix}$

$= \begin{pmatrix} 3 \\ -2 \\ 4 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ -6 \end{pmatrix} - \begin{pmatrix} 0 \\ 15 \\ 3 \end{pmatrix}$

$= \begin{pmatrix} 7 \\ -17 \\ -5 \end{pmatrix}$

$|\tilde{r} - \tilde{q}|$

$= \left| \begin{pmatrix} 0 \\ 5 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix} \right|$

$= \left| \begin{pmatrix} -2 \\ 5 \\ 4 \end{pmatrix} \right|$

$= \sqrt{[(-2)^2 + 5^2 + 4^2]}$

$= \sqrt{45}$

$= \sqrt{9 \times 5}$

$= 3\sqrt{5}$

(10) $$\begin{pmatrix} b \\ 5 \\ -6 \end{pmatrix} = k \begin{pmatrix} 8 \\ -3 \\ a \end{pmatrix} \quad \text{for some } k$$

$$\begin{pmatrix} b \\ 5 \\ -6 \end{pmatrix} = \begin{pmatrix} 8k \\ -3k \\ ak \end{pmatrix}$$

$\therefore$ $b = 8k$ ... (1)

$5 = -3k$ ... (2)

$-6 = ak$ ... (3)

(2): $k = -5/3$

subst. $k = -5/3$ in (3):

$-6 = a \times -5/3$

$a = \frac{18}{5}$

subst. $k = -5/3$ in (1):

$b = 8 \times -5/3$

$b = \frac{-40}{3}$

$\therefore$ $a = \frac{18}{5}$ and $b = \frac{-40}{3}$

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(11) $|e\hat{i} - (e-1)\hat{j}| = 5$

$$\sqrt{[e^2 + \{-(e-1)\}^2]} = 5$$

$$e^2 + (e-1)^2 = 25$$

$$e^2 + e^2 - 2e + 1 = 25$$

$$2e^2 - 2e - 24 = 0$$

$$e^2 - e - 12 = 0$$

$$(e-4)(e+3) = 0$$

$e - 4 = 0$ or $e + 3 = 0$

$e = 4$ or $e = -3$

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(12)

$$\left\| \begin{pmatrix} k+1 \\ 2k \\ k-3 \end{pmatrix} \right\| = \sqrt{12}$$

$$(k+1)^2 + (2k)^2 + (k-3)^2 = 12$$
$$k^2+2k+1+4k^2+k^2-6k+9 = 12$$
$$6k^2 - 4k - 2 = 0$$
$$3k^2 - 2k - 1 = 0$$
$$(3k+1)(k-1) = 0$$
$$3k+1=0 \quad \text{or} \quad k-1=0$$
$$k = -\tfrac{1}{3} \quad \text{or} \quad k = 1$$

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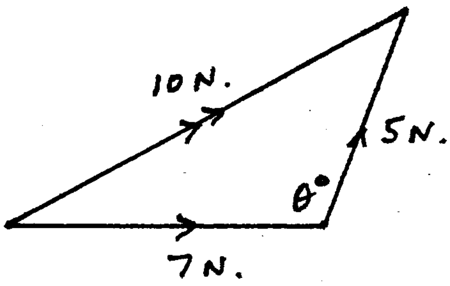
(13)

$$\hat{a}$$
$$= \frac{\tilde{a}}{|\tilde{a}|}$$
$$= \frac{2\hat{i} - 2\hat{j} + \hat{k}}{\sqrt{[2^2+(-2)^2+1^2]}}$$
$$= \frac{2\hat{i} - 2\hat{j} + \hat{k}}{3}$$
$$= \frac{2}{3}\hat{i} - \frac{2}{3}\hat{j} + \frac{1}{3}\hat{k}$$

(18) Consider triangle law of addition of vectors:



by Cosine Rule:

$$\cos\theta = \frac{7^2+5^2-10^2}{2\times 7\times 5}$$

$$\theta = 111{\cdot}8$$

$\therefore$ angle between the two forces

$= 180° - 111{\cdot}8°$

$= 68{\cdot}2°$

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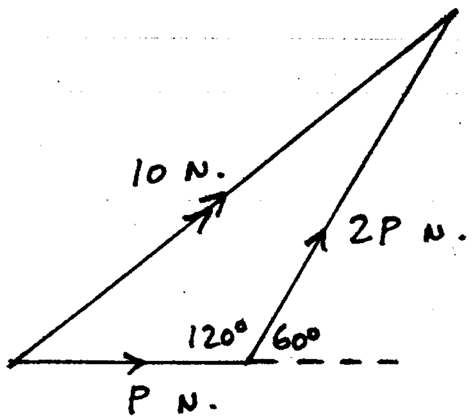
(19) Consider the triangle law of addition of vectors.
Let the forces be P and 2P newtons.



by Cosine Rule:

$$10^2 = P^2 + (2P)^2 - 2\times P\times 2P\times\cos 120°$$

$$100 = P^2 + 4P^2 - 4P^2\times -\tfrac{1}{2}$$

$$100 = 7P^2$$

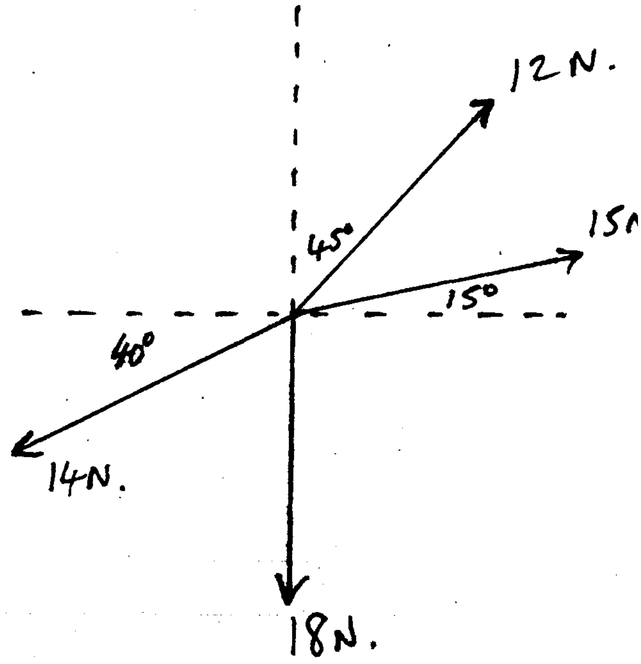
$$P = \sqrt{\frac{100}{7}}$$

$$P = 3{\cdot}77964473$$

$$2P = 7{\cdot}5592896$$

$\therefore$ The forces are $3{\cdot}8$ N and $7{\cdot}6$ N.

(20)



resultant

$$= \begin{pmatrix} 15\cos 15° \\ 15\sin 15° \end{pmatrix} + \begin{pmatrix} 12\cos 45° \\ 12\sin 45° \end{pmatrix} + \begin{pmatrix} 14\cos 220° \\ 14\sin 220° \end{pmatrix} + \begin{pmatrix} 18\cos 270° \\ 18\sin 270° \end{pmatrix}$$

$$= \begin{pmatrix} 12.24954656 \\ -14.63145948 \end{pmatrix}$$

$$= (19.1, -50.1°)$$

$\therefore$ Resultant is 19.1 N acting on a bearing of 140.1°

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(21) Let the unknown force be P N.

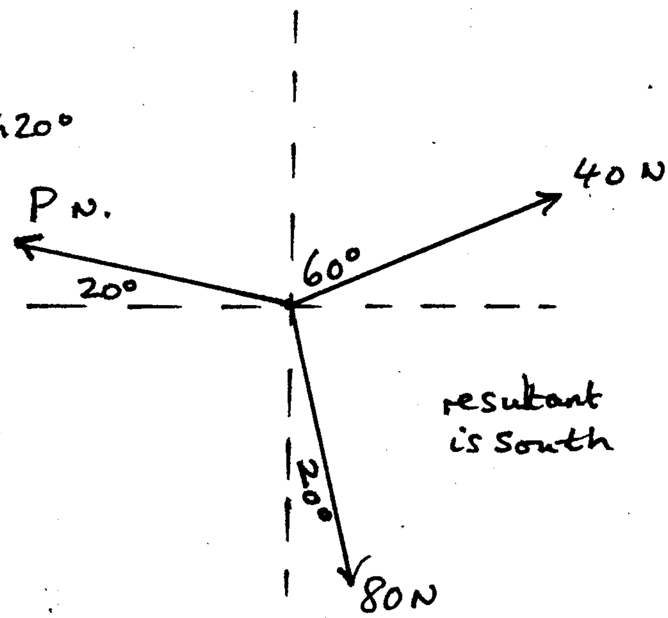
R(W–E direction):

$$P\cos 20° = 40\sin 60° + 80\sin 20°$$

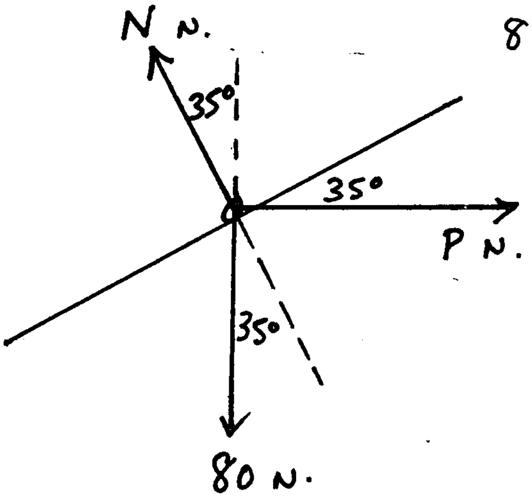
$$P = \frac{40\sin 60° + 80\sin 20°}{\cos 20°}$$

$$P = 65.98181815$$

$\therefore$ magnitude of third force is 66.0 N.



(22)(a)



R(↑): $N\cos 35° = 80$

$N = \dfrac{80}{\cos 35°}$

$N = 97.6619671$

R(→): $P = 97.6619671 \sin 35°$

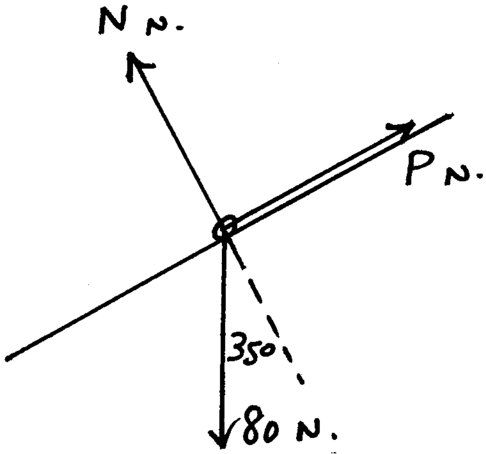
$P = 56.01660306$

∴ P is 56.0 and the normal reaction is 97.7 N.

(b) R(// slope):

$P = 80 \sin 35°$

$P = 45.88611491$



R(⊥ slope):

$N = 80 \cos 35°$

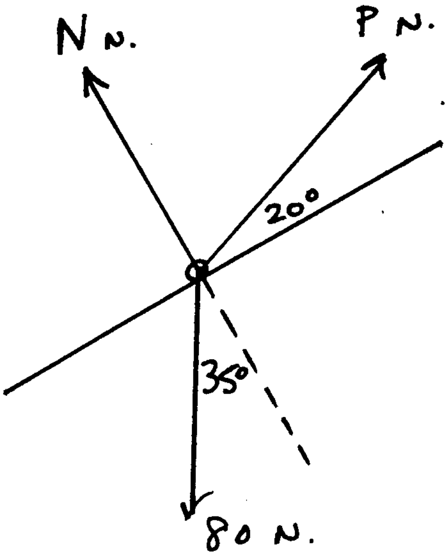
$N = 65.53216354$

∴ P is 45.9 and the normal reaction is 65.5 N.

(c) R(// slope):

$P \cos 20° = 80 \sin 35°$

$P = 48.83098355$



R(⊥ slope):

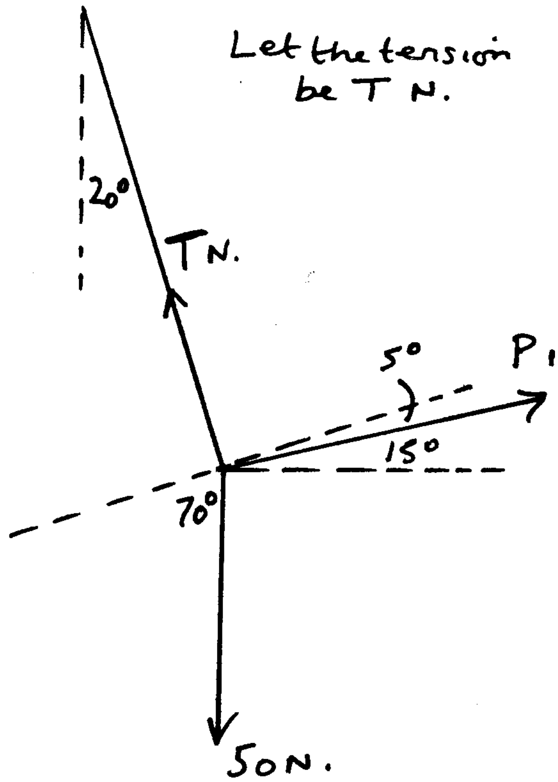
$N + P\sin 20° = 80\cos 35°$

$N = 80\cos 35° - 48.83098355 \sin 20°$

$N = 48.83098355$

∴ P is 48.8 and the normal reaction is 48.8 N.

(23)



R($\perp$ string):

$$P\cos 5^\circ = 50\cos 70^\circ$$
$$P = 17.16633024$$

R(// string):

$$T = P\sin 5^\circ + 50\sin 70^\circ$$
$$T = 17.16633024\sin 5^\circ + 50\sin 70^\circ$$
$$T = 48.4807753$$

$\therefore$ Tension is 48.4 N
and the value of P is 17.2.

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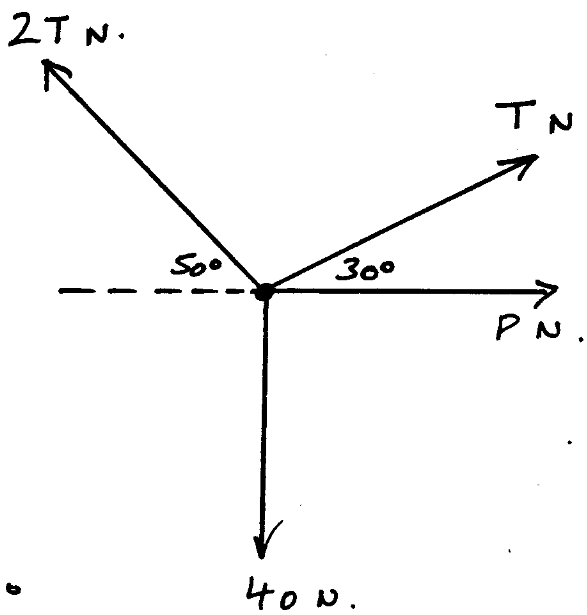
(24)

Let the tensions
be 2T and T newtons.



R(↑):

$$2T\sin 50^\circ + T\sin 30^\circ = 40$$
$$T(2\sin 50^\circ + \sin 30^\circ) = 40$$
$$T = \frac{40}{2\sin 50^\circ + \sin 30^\circ}$$
$$T = 19.68417832$$

R(→):

$$P + T\cos 30^\circ = 2T\cos 50^\circ$$
$$P = 2T\cos 50^\circ - T\cos 30^\circ$$
$$P = T(2\cos 50^\circ - \cos 30^\circ)$$
$$P = 19.68417832 \times (2\cos 50^\circ - \cos 30^\circ)$$
$$P = 8.258493384$$

$\therefore$ P is 8.3