Linear Transformations and Affine Linear Transformations of the Cartesian Plane

Affine transformation - a transformation on a linear space to itself which can be expressed as the sum of a linear transformation and a fixed vector. The affine transformations form a group. In the plane, the group is six-dimensional, consisting of translations, rotations, stretchings and shrinkings, reflections, simple elongations and compressions, and simple shear transformations, as well as compositions of these.
Harcourt Academic Press Dictionary of Science and Technology

An affine linear transformation T is a function that maps each point (x, y) in the cartesian plane to the point (x', y') according to the linear equations:

$$x' = a\,x + b\,y + e$$

$$y' = c\,x + d\,x + f$$

or in matrix form:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a\,x + b\,y + e \\ cx + d\,y + f \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{or} \quad \mathbf{w}' = \mathbf{A}\,\mathbf{w} + \mathbf{b}$$

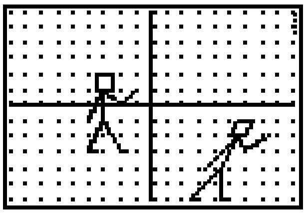
and using function notation:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e \\ f \end{pmatrix} \quad \text{or} \quad T(\mathbf{w}) = \mathbf{A}\,\mathbf{w} + \mathbf{b}$$

An affine linear transformation maps straight lines to straight lines.

Affine linear transformations can be explored by drawing a shape and transforming the shape by selecting values for a, b, c, d, e and f.

The original shape is called the object and the new shape is called the image.



A linear transformation is a special type of affine linear transformation for which $e = 0$ and $f = 0$:

$$x' = a\,x + b\,y$$

$$y' = c\,x + d\,x$$

or in matrix form:

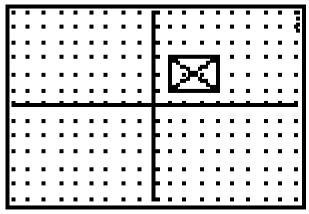
$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a\,x + b\,y \\ cx + d\,y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad \mathbf{w}' = \mathbf{A}\,\mathbf{w}$$

and using function notation:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \quad \text{or} \quad T(\mathbf{w}) = \mathbf{A}\,\mathbf{w}$$

A linear transformation always maps the origin onto itself.

Task 1
Draw the object opposite.



Choose values for a, b, c, d, e and f and draw the image of the object.

Which of the following are properties of affine linear transformations?
a) The midpoint of a line remains the midpoint.
b) The length of a line is unchanged.
c) Parallel lines remain parallel.
d) The angle between two lines is unchanged.
e) The area of a shape is unchanged.
*Could you prove that these properties apply to any object under any affine linear transformation?

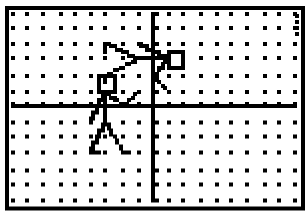
Task 2
What values of a, b, c, d, e and f are needed to achieve the following affine linear transformations? Check your answers by transforming an object.
a) identity transformation
b) translation (shift) of 3 to the right
c) translation of 2 down
d) translation of 5 left and 1 up?

Task 3
The following linear transformations are all reflections. What are the lines of reflection?

a) $\mathbf{A}=\begin{pmatrix}1 & 0\\ 0 & ^{-}1\end{pmatrix}$ b) $\mathbf{A}=\begin{pmatrix}^{-}1 & 0\\ 0 & 1\end{pmatrix}$ c) $\mathbf{A}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$ d) $\mathbf{A}=\begin{pmatrix}0 & ^{-}1\\ ^{-}1 & 0\end{pmatrix}$

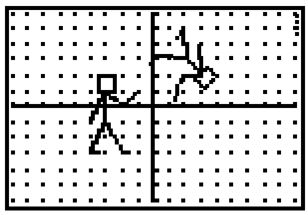
All points on a line of reflection do not move.

Task 4
The following linear transformations are all rotations about the origin. What is the angle and direction of each rotation?



a) $\mathbf{A}=\begin{pmatrix}0 & 1\\ ^{-}1 & 0\end{pmatrix}$ b) $\mathbf{A}=\begin{pmatrix}0 & ^{-}1\\ 1 & 0\end{pmatrix}$ c) $\mathbf{A}=\begin{pmatrix}^{-}1 & 0\\ 0 & ^{-}1\end{pmatrix}$

d) $\mathbf{A}=\begin{pmatrix}\cos 30° & ^{-}\sin 30°\\ \sin 30° & \cos 30°\end{pmatrix}$ e) $\mathbf{A}=\begin{pmatrix}\cos 135° & ^{-}\sin 135°\\ \sin 135° & \cos 135°\end{pmatrix}$



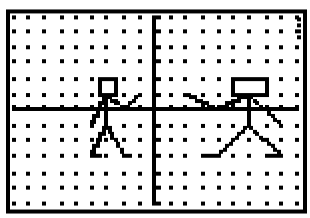
What is the matrix form of this rotation about the origin?

For a rotation about the origin, the origin is the only point that does not move.

Task 5

Linear transformations with matrices in the form $\begin{pmatrix}k & 0\\ 0 & 1\end{pmatrix}$ are dilations in the x direction.

Linear transformations with matrices in the form $\begin{pmatrix}1 & 0\\ 0 & k\end{pmatrix}$ are dilations in the y direction.



Investigate the effect of different values of k, including negative values.

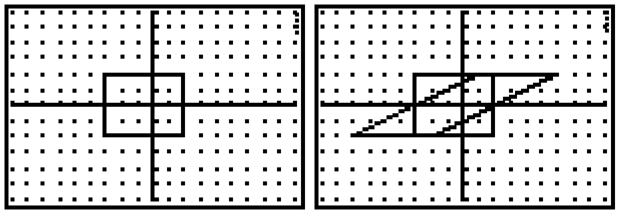
In the case of a dilation in the x direction, all points on the y-axis do not move. For a dilation in the y direction, all points on the x-axis do not move.

Task 6

Linear transformations with matrices in the form $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}$ are called shears.

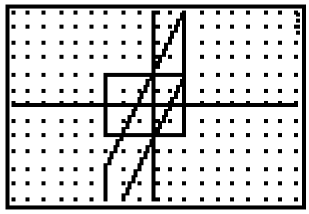
The shear transformation is like tipping a pile of books.

For $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x+2y \\ y \end{pmatrix}$:



- *points on the x-axis do not move*
- other points move in the in the x direction a distance equal to twice the y coordinate

For $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y+2x \end{pmatrix}$:



- *points on the y-axis do not move*
- other points move in the in the y direction a distance equal to twice the x coordinate

Investigate the effect of different values of k, including negative values.

Task 7

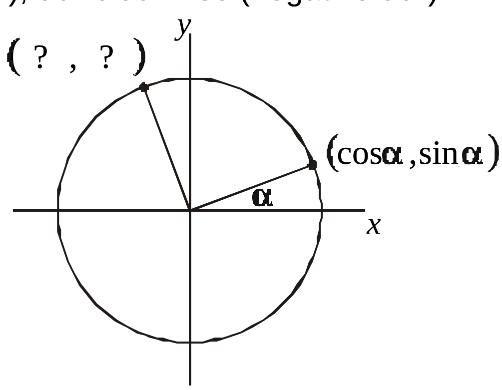
There is an easy method of finding the matrix for a linear transformation. Just consider the images of the points $(1,0)$ and $(0,1)$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix}$ The first column of the matrix is the image of $(1,0)$.

$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix}$ The second column of the matrix is the image of $(0,1)$.

Use this method to write down the matrices for the following linear transformations:

a) reflections in the x-axis, y-axis, $y = x$ and $y = {}^{-}x$
b) rotations about the origin of 90° anti-clockwise (positive 90°), 90° clockwise (negative 90°) and 180°
c) dilations of factor 2, ½ and -3 in the x direction
d) dilations of factor 2, ½ and -3 in the y direction



The matrix for a rotation of angle a about the origin is:

$$\mathbf{A} = \begin{pmatrix} \cos a & {}^{-}\sin a \\ \sin a & \cos a \end{pmatrix}$$

Justify this result by consideration of the unit circle.

Task 8

The matrix for a composite linear transformation can be found by multiplying the individual matrices in the correct order. For example if T_1 is followed by T_2, then the matrix for the composite transformation $T_2 \circ T_1$ is $\mathbf{A}_2\,\mathbf{A}_1$:

$$T_2 \circ T_1(\mathbf{w}) = T_2(T_1(\mathbf{w})) = T_2(\mathbf{A}_1\,\mathbf{w}) = \mathbf{A}_2(\mathbf{A}_1\,\mathbf{w}) = (\mathbf{A}_2\,\mathbf{A}_1)\mathbf{w}$$

Pick two transformations T_1 and T_2 from the following:

- reflections in the x-axis, y-axis, $y = x$ and $y = {}^{-}x$
- rotations about the origin of 90° anti-clockwise (positive 90°), 90° clockwise (negative 90°) and 180°

What is the single transformation that has the same effect as T_1 followed by T_2? Check that the matrix for the single transformation is $\mathbf{A}_2\,\mathbf{A}_1$.

What is the single transformation that has the same effect as T_2 followed by T_1. Check that the matrix for the single transformation is $\mathbf{A}_1\,\mathbf{A}_2$.

Task 9

If a linear transformation T has an inverse, then the matrix of the inverse transformation T^{-1} is the inverse matrix.

$$\mathbf{w}' = \mathbf{A}\,\mathbf{w}$$
$$\mathbf{A}^{-1}\,\mathbf{w}' = \mathbf{w}$$

a) Use the result $\mathbf{A}^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & {}^{-}b \\ {}^{-}c & a \end{pmatrix}$ to check this for each of the different linear transformations - reflections, rotations about the origin, dilations and shears.

b) Use the language of linear transformations to explain the matrix result $(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\,\mathbf{A}^{-1}$.

Task 10

a) Consider the linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Use an inverse matrix to find the point that is mapped onto $({}^{-}1, 7)$.

b) Consider the affine linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & {}^{-}1 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ {}^{-}2 \end{pmatrix}$.

Use an inverse matrix to find the point that is mapped onto $(5, {}^{-}6)$.

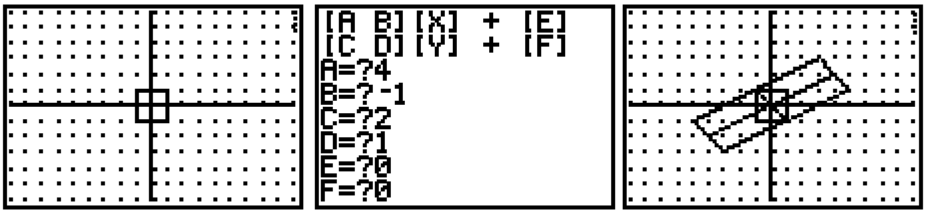
Task 11
Consider an object that is a triangle PQR and an affine linear transformation with determinant $|\mathbf{A}| = ad - bc$. If the vertices P, Q and R appear in clockwise order around the triangle, what is the effect of the affine linear transformation on the order of the vertices if:

a) $|\mathbf{A}| < 0$

b) $|\mathbf{A}| > 0$?

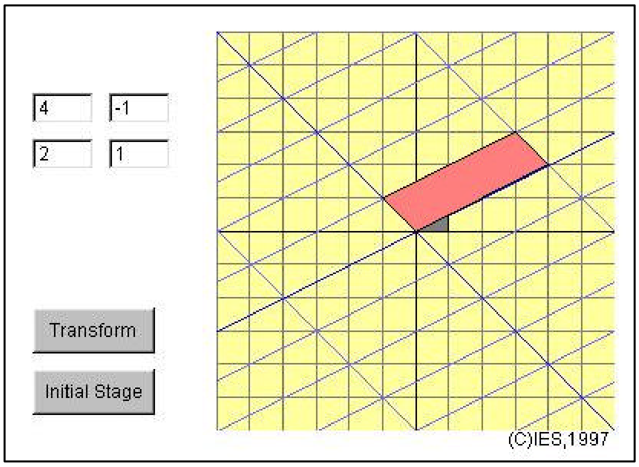
Task 12
Consider the linear transformation:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



One method of exploring the effect of a transformation is to consider the effect on unit squares. If a linear transformation has an inverse, then the grid of squares is transformed into a grid of parallelograms.

A java applet is available to demonstrate this:
http://www.ies.co.jp/math/java/misc/lintra/lintra.html



The area of each parallelogram is equal to the absolute value of the determinant $|\mathbf{A}|$. Check that the area of each parallelogram for this transformation is 6 square units ($|\mathbf{A}| = 4\times 1 - ^{-}1\times 2 = 6$).

Consider some objects under this transformation and show that the areas of the images are increased by a factor of 6.

Investigate the effect of affine linear transformations for which $|\mathbf{A}| = 0$. One possibility would be that each row of **A** is a multiple of the other row. Are there other possibilities?

Task 13
Consider two affine linear transformations:

$$T_1\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} \quad \text{and} \quad T_2\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 & ^{-}3 \\ 5 & ^{-}2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ ^{-}3 \end{pmatrix}$$

a) The composite transformation $T_2 \circ T_1$ means T_1 followed by T_2. Show that $T_2 \circ T_1$ is an affine linear transformation by completing the following:

$$\begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 6 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

$$\searrow$$

$$\begin{pmatrix} 7 & ^{-}3 \\ 5 & ^{-}2 \end{pmatrix}\begin{pmatrix} ? \\ ? \end{pmatrix} + \begin{pmatrix} 4 \\ ^{-}3 \end{pmatrix} = \begin{pmatrix} ? \\ ? \end{pmatrix}$$

b) Is $T_1 \circ T_2$ the same as $T_2 \circ T_1$? Justify your answer.

Task 14

A reflection in the line $y = 3$ is an affine linear transformation. To find the matrix form, perform the composite transformation $T_3 \circ T_2 \circ T_1$:

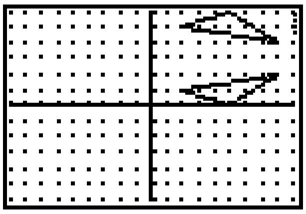
T_1 a translation of 3 down

followed by

T_2 a reflection in the x axis

followed by

T_3 a translation of 3 up.



Complete the following:

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\^-3\end{pmatrix}=\begin{pmatrix}?\\?\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & ^-1\end{pmatrix}\begin{pmatrix}?\\?\end{pmatrix}=\begin{pmatrix}?\\?\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}\begin{pmatrix}?\\?\end{pmatrix}+\begin{pmatrix}0\\3\end{pmatrix}=\begin{pmatrix}?\\?\end{pmatrix}$$

and show that $T_3 \circ T_2 \circ T_1\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}1 & 0\\0 & ^-1\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}+\begin{pmatrix}0\\6\end{pmatrix}$.

Find the matrix form of the following affine linear transformations and check your answers by transforming an object:

a) a reflection in the line $y = {}^-2$

b) a reflection in the line $x = 4$

Task 15

A rotation of 90° clockwise about the point $(3,1)$ is an affine linear transformation. To find the matrix form, perform the composite transformation $T_3 \circ T_2 \circ T_1$:

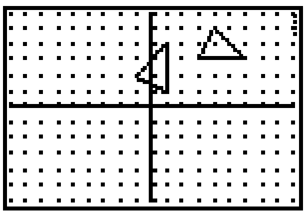
T_1 a translation of 3 left and 1 down

followed by

T_2 a rotation of 90° clockwise about the origin

followed by

T_3 a translation of 3 right and 1 up.



Find the matrix form of the following affine linear transformations and check your answers by transforming an object:

a) a rotation of 90° anti-clockwise about the point $({}^-2,4)$

b) a rotation of 180° about the point $(2, {}^-3)$.

Task 16

Find the matrix form of the shear transformation that has a shear factor $k = 2$ and for which points on the line $x = {}^-3$ do not move. Check your answer by transforming an object.

Task 17*
Any linear transformation with an inverse can be represented by a composition of dilations and shears.

For example, consider the linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 & 3 \\ ^{-}4 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Pre-multiply by a shear matrix that makes the element in the 1st row and 2nd column 0:

$$\begin{pmatrix} 1 & ^{-}\frac{3}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ ^{-}4 & 2 \end{pmatrix}$$

Pre-multiply by a shear matrix that makes the element in the 2nd row and 1st column 0:

$$\begin{pmatrix} 1 & 0 \\ \frac{4}{11} & 1 \end{pmatrix}\begin{pmatrix} 11 & 0 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 11 & 0 \\ 0 & 2 \end{pmatrix}$$

Pre-multiply by a dilation matrix that makes the element in the 1st row and 1st column 1:

$$\begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Pre-multiply by a dilation matrix that makes the element in the 2nd row and 2nd column 1:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Combining these results:

$$\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 11 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 11 & 0 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{4}{11} & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 1 & ^{-}\frac{3}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 3 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{4}{11} & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 5 & 3 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & ^{-}\frac{3}{2} \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ \frac{4}{11} & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{1}{11} & 0 \\ 0 & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}^{-1}$$

$$\begin{pmatrix} 5 & 3 \\ ^{-}4 & 2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ ^{-}\frac{4}{11} & 1 \end{pmatrix}\begin{pmatrix} 11 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Therefore the linear transformation T is equivalent to:

a dilation in the y direction of factor 2

followed by

a dilation in the x direction of factor 11

followed by

a shear in the y direction of factor $^{-}\frac{4}{11}$

followed by

a shear in the x direction of factor $\frac{3}{2}$.

The order of the dilations can be reversed, similarly the order of the shears.

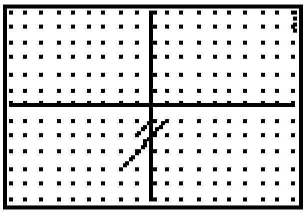
Choose a linear transformation and represent it as a composition of dilations and shears. Check your result by starting with a suitable object and drawing the image after each of the individual transformations.

Task 18*

For the linear transformation $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$, pick any point on the line $y = x$ and show that its image is also on the line $y = x$ i.e. the line is mapped onto itself. Can you see the other line that is mapped onto itself?

A java applet (http://www.ies.co.jp/math/java/misc/koyuve/koyuve.html) demonstrates the effect of a linear transformation on lines that pass through the origin. Lines that are mapped onto themselves are coloured red.

For this linear transformation, any vector in the direction of the line $y = x$ such as $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ does not change direction. It is called an eigenvector:



$$\begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

The value 3 is the eigenvalue corresponding to the eigenvector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

If $\mathbf{w}$ is an eigenvector with matching eigenvalue l, then $\mathbf{A\,w} = l\,\mathbf{w}$. Eigenvalues can be found by solving the equation $\left|\mathbf{A} - l\,\mathbf{I}\right| = 0$ or the equivalent $l^2 - (a + d)\,l + (a\,d - b\,c) = 0$.

Use this equation to show that the eigenvalues for $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ are $l = 3$ and $l = 2$.

Once you have found the eigenvalues, the matching eigenvectors can be found:

$$\mathbf{A}\,\mathbf{w} = l\,\mathbf{w}$$

$$\begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = l\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 4 & ^{-}1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 3\begin{pmatrix} x \\ y \end{pmatrix}$$

$$\begin{pmatrix} 4x - y \\ 2x + y \end{pmatrix} = \begin{pmatrix} 3x \\ 3y \end{pmatrix}$$

which is true if $y = x$ i.e. $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is the eigenvector corresponding to the eigenvalue 3.

Use this method to find the eigenvector corresponding to the eigenvalue 2.

Draw a triangle with two of its sides parallel to the eigenvectors. What do you notice about the sides of the image triangle?

Investigate the eigenvalues and eigenvectors of other linear transformations, including examples of linear transformations that have complex eigenvalues.

Task 19*
By considering an appropriate composite transformation, find the matrix form of each of the following affine linear transformations:
a) reflection in the line $y = x + 3$
b) reflection in the line $y = 2x$
c) reflection in the line $y = 2x + 3$.

Task 20*
Under a linear transformation, the vertices $(0,0)$, $(1,0)$, $(0,1)$ and $(1,1)$ of the unit square are mapped to points as follows:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} a \\ c \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} b \\ d \end{pmatrix} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$$

Show that these points form a parallelogram and that the area of the parallelogram is $|ad - bc|$ square units.

Task 21*
Linear transformations have the special properties:

$$T(\mathbf{w} + \mathbf{v}) = T(\mathbf{w}) + T(\mathbf{v})$$
$$T(k\mathbf{w}) = kT(\mathbf{w})$$

Use a counter example to show that these properties are not held by all affine linear transformations.

Task 22*

Parabolas remain parabolas under affine linear transformations. Investigate.

Task 23*

Affine linear transformations of two dimensional space can be represented by 3 x 3 matrices:

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & e \\ c & d & f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

Investigate composition of transformations and inverse transformations using this representation.

Task 24*

Investigate Iterated Function Systems. Only four affine linear transformations are needed to produce the fractal fern opposite i.e. the picture is defined by exactly 24 numbers (the four sets of a, b, c, d, e and f).

http://astronomy.swin.edu.au/pbourke/fractals/fern/